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A Direct Study in a Hilbert-Schmidt Framework of the Riccati Equation Appearing in a Factorization Method of Second Order Elliptic Boundary Value Problems*

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Abstract: In this report we come back to the method of factorization of a second order elliptic boundary value problem presented in [7]. In this paper, it was shown that, in the case of a cylinder, the boundary value problem can be factorized in two uncoupled first order initial value problems. This factorization utilizes the Dirichlet to Neumann operator which satisfies a Riccati equation. Here we consider Hilbert-Schmidt operators, a framework already used by R. Temam [12] which provides tools for a direct study of this Riccati equation.

Key-words: factorization, boundary value problem, Dirichlet-Neumann operator, Hibert-Schmidt operator, Riccati equation

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Sur une étude directe dans un cadre de Hilbert-Schmidt de l'équation de Riccati intervenant dans la factorisation d'un problème aux limites elliptique du second ordre

Résumé : Dans ce rapport on revient la méthode de factorisation d'un problème aux limites elliptique du second ordre présenté dans [7]. Dans ce papier on avait en effet montré que, dans le cas d'un domaine cylindrique, le problème aux limtes peut se factoriser en deux problèmes aux valeurs initiales du premier ordre non couplés. Cette factorisation fait intervenir l'opérateur Dirichlet-Neumann qui vérifie une équation de Riccati. On se place ici dans le cadre des opérateurs de Hilbert-Schmidt en se référant au travail de R. Temam [12]. Ce cadre permet une étude directe de cette équation de Riccati.

Mots-clés : factorisation, problème aux limites, opérateur Dirichlet-Neumann, opérateur de Hibert-Schmidt, équation de Riccati

Contents

1 Introduction

In [10], [7] we developed a factorization method for second order elliptic boundary value problems by dynamic programming. The principle of the method was presented in the book by Angel and Bellman [3]. We restrict the presentation to the simple geometrical case where the domain is a cylinder whose axis is parallel to the x_1 coordinate. This coordinate will play a distinguished role. Namely, let $\mathcal{O}$ be a bounded open set in $\mathbb{R}^{n-1}$, Ω be the cylinder $\Omega =]0,a[\times\mathcal{O}$ in $\mathbb{R}^n$ and $\Gamma_s = \{s\}\times\mathcal{O}$. The lateral boundary of the cylinder is denoted by $\Sigma = \partial\mathcal{O}\times]0,a[$. Let $f\in L^2(\Omega)$ and $y_0, y_a \in H_{00}^{1/2}(\mathcal{O})$. Let us denote $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} = \frac{\partial^2}{\partial x_1^2} + \Delta_z$, where z denotes the independent variables $x_2, ..., x_n$. Let us consider the problem

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \\ y = y_0 & \text{on } \Gamma_0, \\ \dfrac{\partial y}{\partial x_1} = y_a & \text{on } \Gamma_a \quad (\text{respectively } y = y_a \text{ on } \Gamma_a). \end{cases} \tag{1}$$

In [10], [7] we showed that this problem can be factorized as

$$\begin{cases} \dfrac{dP}{dx_1} + P\Delta_z P + I = 0, \quad P(0) = 0, \\ \dfrac{dr}{dx_1} + P\Delta_z r = -Pf, \quad\ r(0) = y_0, \\ P\dfrac{dy}{dx_1} + y = r, \quad y(a) = -P(a)y_a + r(a) \ (\text{respectively } y(a) = y_a). \end{cases} \tag{2}$$

Similarly, problem (1) (respectively $-\dfrac{\partial y}{\partial x_1} = y_0$ on Γ_0), can be factorized as

$$\begin{cases} \dfrac{dQ}{dx_1} - Q^2 - \Delta_z = 0, \quad Q(a) = 0, \\ \dfrac{dw}{dx_1} - Qw = f, \qquad w(a) = y_a, \\ \dfrac{dy}{dx_1} + Qy = -w, y(0) = y_0 \ (\text{respectively } y(0) = Q^{-1}(0)(y_0 - w(0))). \end{cases} \tag{3}$$

In this article we shall study the direct solution of system (2). A similar study for the system (3) shall be carried out in a further article.

In [10], [7] we showed a justification of the derivation of the Riccati equation

$$\begin{cases} \dfrac{dP}{dx_1} + P\Delta_z P + I = 0, \\ P(0) = 0. \end{cases} \tag{4}$$

We obtained its solution in a general distribution sense and we showed some regularity results. All this was done by using the fact that the operator P was defined as the Neumann-to-Dirichlet operator on the boundary of a subdomain (following the invariant imbedding techniques developed by Bellman). We used a Galerkin method to study the problem in finite dimension. The passing to the limit for the infinite dimensional problem was similar to the method used by J.L. Lions in [8] to derive the optimal feedback of linear quadratic control of parabolic problems. It was also shown that this factorization could be viewed as an extension to the infinite dimensional problem of the block Gauss LU factorization.

In this article we follow the inverse way, that is, we first study directly the solutions of the Riccati equation (4) and then we show that the solution of this equation is the operator relating the Dirichlet and Neumann conditions of the same second order elliptic boundary value problem as before. This direct method for studying the Riccati equation provides a more efficient way of deriving the factorized form (1) of (2). It is hoped to be helpful for the case of more complex (non cylindrical) domains and more general differential operators. We shall give much more detailed results for the solution of (4) than those given in [10] and [7].

In [12] R. Temam studied the equation

$$\left\{\begin{array}{l} \dfrac{dP}{dx_1} + A^* \circ P + P \circ A + P \circ P = F, \\ P(0) = P_0, \end{array}\right.$$

in a Hilbert-Schmidt space framework. In the above equation, A is a coercive operator, which allowed the author to get a solution $P \in \mathcal{C}([0,T] : H\widehat{\otimes}_2 H)$, with H being a suitable Hilbert space. In equation (4) there is not an equivalent coercive operator A and therefore, we are not able to use the theory developed in [12]. We shall justify equation (4), also in a Hilbert-Schmidt space framework, by using the Galerkin method. We shall use the notation $\varphi_1 \otimes \varphi_2$ and $X\widehat{\otimes}_2 Y$, in the context of Hilbert-Schmidt spaces, as used in [12].

In Section 2 we introduce the functional space framework we are going to deal with. In Section 3 we develop the direct study of the Riccati equation appearing in (2). Finally, in Section 4 we show how to obtain the solution of different elliptic equations with the help of the factorization method, pointing out that this method is well suited for setting transparent boundary conditions.

2 Hilbert-Schmidt Spaces

2.1 Generalities

In this section we remind the definitions and properties of Hilbert-Schmidt spaces (see [5] and [1]) following the notation given in [12]. We consider two Hilbert spaces X, Y with scalar products $(\cdot,\cdot)_X$, $(\cdot,\cdot)_Y$ and associated norms $\|\cdot\|_X$, $\|\cdot\|_Y$. We suppose that X and Y are separable and $\{\varphi_i\}_{i=1}^{\infty}$ $\{\psi_i\}_{i=1}^{\infty}$ are any orthonormal basis of X and Y respectively.

Definition 1 $P \in \mathcal{L}(X,Y)$ *(a bounded linear operator from X to Y) is said to be a Hilbert-Schimdt operator in case*

$$\sum_{i=1}^{\infty} \|P(\varphi_i)\|_Y^2 < \infty.$$

In this case we denote $P \in X \widehat{\otimes}_2 Y$ and

$$\|P\|_{X\widehat{\otimes}_2 Y} = \left(\sum_{i=1}^{\infty} \|P(\varphi_i)\|_Y^2 \right)^{1/2}$$

is called the Hilbert-Schmidt norm of P.

The proof of the following properties and other results regarding Hilbert-Schmidt spaces can be seen in [5] and [1].

Lemma 1 *The Hilbert-Schmidt norm is independent of the orthonormal basis used in its definition and*

$$\|P\|_{X\widehat{\otimes}_2 Y} = \left(\sum_{i,j=1}^{\infty} |(P\varphi_i, \psi_j)_Y|^2 \right)^{1/2}.$$

Proposition 1 *Every Hilbert-Schmidt operator is compact.*

Proposition 2 *$X\widehat{\otimes}_2 Y$ is a Hilbert space with the inner product*

$$[P, Q]_{X\widehat{\otimes}_2 Y} = \sum_{i=1}^{\infty} (P\varphi_i, Q\varphi_i)_Y.$$

2.2 A special case

Following [9], let us suppose that H, V are two real separable Hilbert spaces such that $V \subset H$ and V is dense in H with continuous injection. Then, we can identify H' (topological dual of H) with H and with a dense subset of V'. Therefore,

$$V \subset H \subset V',$$

with every space being dense in the following one with continuous injections.

Let us suppose that the injection $V \subset H$ is compact and let $\Lambda : V \to V'$ be the canonical isomorphism between V and V', i.e.

$$(\Lambda u, v)_{V' \times V} = (u, v)_V \quad \forall\ u, v \in V.$$

Then, $\Lambda^{-1} : H \to V$ is a compact self-adjoint linear operator. Therefore, there exists an orthonormal basis $\{w_i\}_{i=1}^{\infty}$ of H, consisting of eigenvectors of Λ^{-1} such that

$$\Lambda w_i = \lambda_i w_i, \quad \forall\ i \in I\!N$$

with $\lambda_i > 0$ and

$$\lim_{i \to \infty} \lambda_i = +\infty.$$

Now,

$$\|w_i\|_V^2 = (w_i, w_i)_V = (\Lambda w_i, w_i)_{V' \times V} = (\lambda_i w_i, w_i)_H = \lambda_i \quad \forall\, i = 1, \cdots \infty.$$

Therefore,

$$u = \sum_{i=1}^{\infty} u_i w_i \in V \Leftrightarrow \|v\|_V^2 = \sum_{i=1}^{\infty} \lambda_i |u_i|^2 < \infty.$$

Furthermore, if $u = \sum_{i=1}^{\infty} u_i w_i \in V$ and $v = \sum_{i=1}^{\infty} v_i w_i \in V$, then

$$(u, v)_V = \sum_{i=1}^{\infty} \lambda_i u_i v_i.$$

Definition 2 *For every $r \in I\!R$ we define the Hilbert space V^r in the following way:*

$$u = \sum_{i=1}^{\infty} u_i w_i \in V^r \Leftrightarrow \|v\|_{V^r}^2 = \sum_{i=1}^{\infty} \lambda_i^r |u_i|^2 < \infty.$$

Furthermore, if $u = \sum_{i=1}^{\infty} u_i w_i \in V^r$ and $v = \sum_{i=1}^{\infty} v_i w_i \in V^r$, then

$$(u, v)_{V^r} = \sum_{i=1}^{\infty} \lambda_i^r u_i v_i.$$

Remark 1 *We have that $H = V^0$, $V = V^1$ and $V' = V^{-1}$. Furthermore, $\{\lambda_i^{-r/2} w_i\}_{i=1}^{\infty}$ is an orthonormal basis of V^r, for all $r \in I\!R$.*

Definition 3 *Let $\mathcal{H} = \{ \sum_{finite} \mu_j w_j,\ \mu_j \in I\!R\}$ and $< w_j >= \{\mu w_j,\ \mu \in I\!R\}$ for all $j \in I\!N$. Given $i, j \in I\!N$, we consider the operator $w_i \otimes w_j : \mathcal{H} \to < w_j >$ defined by*

$$w_i \otimes w_j(\varphi) = (w_i, \varphi)_H w_j \quad \forall \varphi \in \mathcal{H}.$$

Now, since $\mathcal{H}$ is a dense subset of V^r, for all $r \in I\!R$, we can consider the extension $w_i \otimes w_j : V^r \to < w_j >$ defined by

$$w_i \otimes w_j(\varphi) = \varphi_i w_j \quad \forall \varphi = \sum_{i=1}^{\infty} \varphi_i w_i \ \in V^r.$$

Remark 2 *From Definition 1 it is easy to see that, for all $i, j \in I\!R$, we have that $w_i \otimes w_j \in V^r \widehat{\otimes}_2 V^s$ for all $r, s \in I\!R$. Furthermore, if $P = \sum_{i,j=1}^{\infty} \xi_{ij} w_i \otimes w_j$, then*

$$P \in V^r \widehat{\otimes}_2 V^s \Leftrightarrow \|P\|_{V^r \widehat{\otimes}_2 V^s}^2 = \sum_{i=1}^{\infty} \|P(\lambda_i^{-r/2} w_i)\|_{V^s}^2 < \infty.$$

Now,

$$P(\lambda_i^{-r/2} w_i) = \sum_{k,j=1}^{\infty} \xi_{kj} (w_k, \lambda_i^{-r/2} w_i)_H w_j = \sum_{j=1}^{\infty} \xi_{ij} \lambda_i^{-r/2} w_j.$$

Therefore

$$\|P(\lambda_i^{-r/2} w_i)\|_{V^s}^2 = \sum_{j=1}^{\infty} \xi_{ij}^2 \lambda_i^{-r} \lambda_j^s$$

and we deduce that

$$P \in V^r \widehat{\otimes}_2 V^s \Leftrightarrow \|P\|_{V^r \widehat{\otimes}_2 V^s}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_i^{-r} \lambda_j^s < \infty.$$

Finally, if $Q = \sum_{i,j=1}^{\infty} \eta_{ij} w_i \otimes w_j \in V^r \widehat{\otimes}_2 V^s$*, then*

$$[P,Q]_{V^r \widehat{\otimes}_2 V^s} = \sum_{i=1}^{\infty} (P(\lambda_i^{-r/2} w_i), Q(\lambda_i^{-r/2} w_i))_{V^s} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_i^{-r} \lambda_j^s.$$

Example 1 *Let* $P = \sum_{i,j=1}^{\infty} \xi_{ij} w_i \otimes w_j$ *and* $Q = \sum_{i,j=1}^{\infty} \eta_{ij} w_i \otimes w_j$.

1. $P \in H \widehat{\otimes}_2 H \Leftrightarrow \|P\|_{H \widehat{\otimes}_2 H}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 < \infty$ *and*
$$[P,Q]_{H \widehat{\otimes}_2 H} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij}.$$
2. $P \in V' \widehat{\otimes}_2 H \Leftrightarrow \|P\|_{V' \widehat{\otimes}_2 H}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_i < \infty$ *and*
$$[P,Q]_{V' \widehat{\otimes}_2 H} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_i.$$
3. $P \in V \widehat{\otimes}_2 H \Leftrightarrow \|P\|_{V \widehat{\otimes}_2 H}^2 = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_i^{-1} < \infty$ *and*
$$[P,Q]_{V \widehat{\otimes}_2 H} = \sum_{i,j=1}^{\infty} \xi_{ij} \eta_{ij} \lambda_i^{-1}.$$

4. $P \in H\widehat{\otimes}_2 V \Leftrightarrow \|P\|^2_{H\widehat{\otimes}_2 V} = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_j < \infty$ *and*

$$[P,Q]_{H\widehat{\otimes}_2 V} = \sum_{i,j=1}^{\infty} \xi_{ij}\eta_{ij}\lambda_j.$$

5. $P \in H\widehat{\otimes}_2 V' \Leftrightarrow \|P\|^2_{H\widehat{\otimes}_2 V'} = \sum_{i,j=1}^{\infty} \xi_{ij}^2 \lambda_j^{-1} < \infty$ *and*

$$[P,Q]_{H\widehat{\otimes}_2 V'} = \sum_{i,j=1}^{\infty} \xi_{ij}\eta_{ij}\lambda_j^{-1}.$$

2.3 Our particular case

In this work we shall consider the spaces $V^r\widehat{\otimes}_2 V^s$, $r,s \in \mathbb{R}$, for the particular case of $H = L^2(\mathcal{O})$ and $V = H_0^1(\mathcal{O})$. Therefore, we consider as orthonormal basis of V the set $\{w_1, ..., w_n, ..\}$ of eigenfunctions of the Dirichlet problem $-\Delta_z w_n = \lambda_n w_n$ for $z \in \mathcal{O}$ with the boundary conditions $w_n|_{\partial\mathcal{O}} = 0$. It has the following properties:

$$\begin{cases} (a)\ (w_n, w_m)_{L^2(\mathcal{O})} = \delta_{n,m} \quad \forall m, n. \\ (b)\ (w_n, w_m)_{H_0^1(\mathcal{O})} = \int_{\mathcal{O}} \nabla_z w_n(z) \nabla_z w_m(z) dz = \lambda_n \delta_{n,m}, \\ (c) \{ \sum_{\text{finite}} \mu_j w_j,\ \mu_j \in \mathbb{R} \} \text{ is a dense subset of } V. \end{cases}$$

If we write the sequence in a nondecreasing way, it can be proved that $\lambda_i \geq 0$, for all $i = 1, \cdots, +\infty$ and

$$\lim_{i\to\infty} \lambda_i = +\infty.$$

Proposition 3 *The identity operator I can be expressed as*

$$I = \sum_{i=1}^{\infty} w_i \otimes w_i,$$

with this sequence converging in $\mathcal{L}(H,H)$.

Proof. Let $\varphi \in H$, then

$$\sum_{i=1}^{\infty} w_i \otimes w_i(\varphi) = \sum_{i=1}^{\infty} (\varphi, w_i)_H w_i = \sum_{i=1}^{\infty} \varphi_i w_i = \varphi,$$

where φ_i, $i = 1, \cdots, \infty$ are the coordinates with respect to the orthonormal basis $\{w_i\}_{i=1}^{\infty}$ in $L^2(\mathcal{O})$. □

Lemma 2 (Weyl's Estimate) *Given a regular domain $\Omega \subset I\!R^n$, the asymptotic behavior of the eigenvalues for the Laplace operator with homogeneous Dirichlet boundary conditions is the following:*

$$\lambda_k \sim \frac{4\pi}{(Vol(\Omega))^{2/n}} k^{2/n}.$$

Proof. See [13]. $\square$

We point out that $w_i \in V^s$ for all $s \in I\!R$ and $i \in I\!N$. Further, $V^0 = H$ and $V^1 = V$. Depending on the space dimension, we shall need slightly more regular spaces than H and V.

Definition 4 *Let $\delta = 0$ if $dim(\mathcal{O}) = 1$ and $\delta > 0$ if $dim(\mathcal{O}) = 2$ ($dim(\mathcal{O}) > 2$ is not of interest in applications). Let us define $\widetilde{H} = V^\delta$ and $\widetilde{V} = V^{1+\delta}$ (if $dim(\mathcal{O}) = 1$, then $\widetilde{H} = H$ and $\widetilde{V} = V$). For $\delta \geq 0$ small enough, we have $\widetilde{V} \subset V \subset \widetilde{H} \subset H$.*

Corollary 1 *If $dim(\mathcal{O}) \leq 2$, then $I \in (\widetilde{V} \widehat{\otimes}_2 H) \cap (H \widehat{\otimes}_2 \widetilde{V}')$.*

Proof. We have to prove that

$$\sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} < +\infty,$$

which is a consequence of Lemma 2. $\square$

3 A direct Study of the Riccati equation (4)

3.1 Semi discretization

3.1.1 Formulation of the problem and monotonicity results

We approximate $\widetilde{V}$, V and H by $V^m = span(w_1, \cdots, w_m)$ and define the m-approximate solution of (4) by the finite dimensional operator

$$P^m(x_1) = \sum_{i,j=1}^{m} \xi_{ij}^m(x_1) w_i \otimes w_j \in V^m \otimes V^m,$$

where $\xi_{ij}^m(x_1)$ are chosen such that

$$\begin{cases} \left[\dfrac{dP^m}{dx_1} + P^m \Delta_z P^m + I^m, w_i \otimes w_j\right]_{H \widehat{\otimes}_2 H} = 0, \qquad \forall\, i,j \in \{1, \cdots m\} \\ P^m(0) = 0, \end{cases} \tag{5}$$

with $I^m = \sum_{i=1}^m w_i \otimes w_i$. We point out that $\xi_{ij}^m(x_1) = (P^m(x_1) w_i, w_j)_H$.

System (5) is a nonlinear system of equations in ξ^m_{ij} of the form:

$$\left\{ \begin{array}{l} \dfrac{d\xi^m}{dx_1} + A^m(\xi^m) + b^m = 0, \\ \xi^m(0) = 0, \end{array} \right.$$

where

$$A^m(\xi^m)_{ij} = \left[\left(\sum_{k,l=1}^{m} \xi^m_{kl} w_k \otimes w_l \right) \Delta_z \left(\sum_{r,s=1}^{m} \xi^m_{rs} w_r \otimes w_s \right), w_i \otimes w_j \right]_{H\widehat{\otimes}_2 H}$$

and

$$b^m_{ij} = \left\{ \begin{array}{ll} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{array} \right.$$

But this derivation is still formal since we have assumed the derivability of P^m. Then, by the theory of ordinary differential equations, we know that there exists a local solution P^m to (5) in $[0, \delta]$, with δ small enough. Further, P^m is $\mathcal{C}^1$ from $[0, \delta]$ with values in $H\widehat{\otimes}_2 H$.

To go further, we need estimates on $\xi^m(s)$ (i.e. on $P^m(s)$) independent of s.

Now, it is easy to verify that

$$A^m(\xi^m)_{ij} = \left[\left(\sum_{k,l=1}^{m} \xi^m_{kl} w_k \otimes w_l \right) \left(\sum_{r,s=1}^{m} -\lambda_s \xi^m_{rs} w_r \otimes w_s \right), w_i \otimes w_j \right]_{H\widehat{\otimes}_2 H}.$$

Now, taking into account that

$$w_k \otimes w_l \circ w_r \otimes w_s = \left\{ \begin{array}{ll} w_r \otimes w_l, & \text{if } k = s, \\ 0, & \text{otherwise,} \end{array} \right.$$

we have that

$$A^m(\xi^m)_{ij} = \left[\sum_{k,l,r=1}^{m} -\lambda_k \xi^m_{kl} \xi^m_{rk} w_r \otimes w_l, w_i \otimes w_j \right]_{H\widehat{\otimes}_2 H} = \sum_{k=1}^{m} -\lambda_k \xi^m_{ik} \xi^m_{kj}.$$

Then, if we write the (i, j)-coordinates in $m \times m$ matrix form, we obtain

$$\left\{ \begin{array}{l} \dfrac{d\xi^m}{dx_1} - \xi^m \Lambda^m \xi^m + I^m = 0, \\ \xi^m(0) = 0, \end{array} \right. \tag{6}$$

where I^m represents here the $m \times m$ identity matrix and Λ^m is a diagonal matrix, with λ_i, $i = 1, \cdots m$, being the elements of the diagonal.

Now, following page 11 of [11], the solution of the above problem can be expressed as

$$\xi^m(x_1) = V(x_1)U(x_1)^{-1},$$

where $Y = \begin{pmatrix} U \\ V \end{pmatrix}$ satisfies the corresponding linear (Hamiltonian) matrix differential system

$$\begin{cases} \mathcal{J}Y' + \mathcal{U}Y = 0 \\ Y(0) = \begin{pmatrix} I^m \\ 0 \end{pmatrix}, \end{cases} \tag{7}$$

with

$$\mathcal{J} = \begin{pmatrix} 0 & -I^m \\ I^m & 0 \end{pmatrix}, \qquad \mathcal{U} = \begin{pmatrix} -I^m & 0 \\ 0 & \Lambda^m \end{pmatrix}.$$

Therefore

$$Y(x_1) = e^{-Ax_1} \cdot \begin{pmatrix} I^m \\ 0 \end{pmatrix}, \qquad \forall\, x_1 \in [0, +\infty),$$

with

$$A = \mathcal{J}^{-1}\mathcal{U} = \begin{pmatrix} 0 & I^m \\ -I^m & 0 \end{pmatrix} \begin{pmatrix} -I^m & 0 \\ 0 & \Lambda^m \end{pmatrix} = \begin{pmatrix} 0 & \Lambda^m \\ I^m & 0 \end{pmatrix}.$$

The eigenvalues of A are $\sqrt{\lambda_i}$ and $-\sqrt{\lambda_i}$, $i = 1, \cdots m$, and the matrix of eigenvectors is

$$\begin{pmatrix} (\Lambda^m)^{1/2} & -(\Lambda^m)^{1/2} \\ I^m & I^m \end{pmatrix}.$$

Then U and V are given by

$$U(x_1) = \frac{1}{2}\left(e^{(\Lambda^m)^{1/2}x_1} + e^{-(\Lambda^m)^{1/2}x_1}\right), \tag{8}$$

$$V(x_1) = \frac{(\Lambda^m)^{-1/2}}{2}\left(e^{-(\Lambda^m)^{1/2}x_1} - e^{(\Lambda^m)^{1/2}x_1}\right). \tag{9}$$

Thus, U is non-singular and we can extend the solution of 5 from $[0, \delta]$ to $[0, +\infty]$. The uniqueness is a consequence of Corollary 2 of page 13 of Reid.

Remark 3 The matrix A has positive and negative eigenvalues. The initial value problem for the Hamiltonian system (7) is well posed in finite dimension. This would not be the case for the corresponding infinite dimensional initial value problem because of the part of system (7) corresponding to negative eigenvalues $\{-\sqrt{\lambda_i}\}_{i=0}^{\infty}$ which are going to $-\infty$. For this part of the system we have a situation similar to a backward heat equation whose ill-posedness is well-known.

Proposition 4 *For a given $m \in I\!N$, we have that, for all $x_1 \geq 0$,*

$$P^m(x_1) = \sum_{i=1}^{m} \xi_{ii}^m(x_1) w_i \otimes w_i,$$

and there exists $P_\infty^m \in V^m \times V^m$ such that, if $0 \leq t_1 \leq t_2$, then

$$0 \geq P^m(t_1) \geq P^m(t_2) \geq P_\infty^m.$$

3.1.2 Analytical Proof of Proposition 4

From (8), (9) the unique solution of (5) is diagonal and given by

$$\xi_{ii}(x_1) = \frac{-1}{\sqrt{\lambda_i}} \left(\frac{e^{\sqrt{\lambda_i}x_1} - e^{-\sqrt{\lambda_i}x_1}}{e^{\sqrt{\lambda_i}x_1} + e^{-\sqrt{\lambda_i}x_1}} \right) = \frac{-1}{\sqrt{\lambda_i}} \tanh(\sqrt{\lambda_i}x_1), \qquad i = 1, \dots, m.$$

It can be computed also as the solution of the diagonal system

$$\left\{ \begin{array}{l} \dfrac{d\xi_{ii}}{dx_1} - \lambda_i \xi_{ii}^2 + 1 = 0, \\ \xi_{ii}(0) = 0. \end{array} \right. \qquad i = 1, \dots, m. \tag{10}$$

Therefore,

$$\frac{d\xi_{ii}}{dx_1}(x_1) = \frac{-1}{\cosh^2(\sqrt{\lambda_i}x_1)} < 0, \quad \forall\, x_1 \in I\!R, \tag{11}$$

and we can define

$$\xi_{ii,\infty}^m := \lim_{x_1 \to \infty} \xi_{ii}(x_1) = \frac{-1}{\sqrt{\lambda_i}} \qquad i = 1, \dots, m,$$

which concludes the proof. $\square$

Remark 4 *From (11) and* $0 > \frac{d\xi_{ii}}{dx_1}(x_1) \geq -1$ *we obtain that*

$$0 \leq \lambda_i \xi_{ii}^2(x_1) < 1, \quad \forall\, x_1 \in I\!R, \quad i = 1, \dots, \infty. \tag{12}$$

Remark 5 The Hamiltonian system (7) is related to an optimal control problem. Let us show it in a componentwise fashion. Let $c \in I\!R$ and $\mathcal{U}_c = \{\varphi \in H^1(0,a) : \varphi(0) = c\}$. Let us consider the problem

$$\left\{ \begin{array}{l} \text{Find } \eta \in \mathcal{U}_c, \text{ such that} \\ J(\eta) \leq J(\varphi), \ \forall\, \varphi \in \mathcal{U}_c, \end{array} \right. \tag{13}$$

where

$$J(\varphi) = \int_0^a \left(\frac{1}{\lambda_i} \varphi'^2 + \varphi^2 \right) dx_1.$$

Then, if we define

$$f(c,a) = \min_{\varphi \in \mathcal{U}_c} \int_0^a \left(\frac{1}{\lambda_i} \varphi'^2 + \varphi^2 \right) dx_1,$$

following the ideas developed in [2], it can be proved (see Appendix A.1) that

$$f(c,a) = -c^2 \xi_{ii}(a),$$

and the minimum is obtained for the function satisfying the Euler equation

$$\left\{ \begin{array}{l} \eta'' - \lambda_i \eta = 0 \\ \eta(0) = c \\ \eta'(a) = 0, \end{array} \right. \tag{14}$$

which is equivalent to the optimality system

$$\begin{cases} \eta' = p; & \eta(0) = 0 \\ -p' = -\lambda_i \eta; & p(a) = 0, \end{cases}$$

and has a unique solution given by

$$\eta(x_1) = \frac{e^{\sqrt{\lambda_i}(a-x_1)} + e^{-\sqrt{\lambda_i}(a-x_1)}}{e^{\sqrt{\lambda_i}a} + e^{-\sqrt{\lambda_i}a}} c = \frac{\cosh(\sqrt{\lambda_i}(a-x_1))}{\cosh(\sqrt{\lambda_i}a)} c.$$

Problem (13) is equivalent to the optimal control problem

$$\begin{cases} \text{Find } u \in L^2(0,a), \text{ such that} \\ \overline{J}(u) \leq \overline{J}(v), \ \forall\ v \in L^2(0,a), \end{cases}$$

where

$$\overline{J}(\varphi) = \int_0^a \left(\frac{1}{\lambda_i} v^2 + \varphi^2 \right) dx_1,$$

with

$$\begin{cases} \varphi' = v \\ \varphi(0) = c. \end{cases}$$

Then, following the ideas developed in [2], it can be proved by Dynamic Programming (see Appendix A.1) that the optimal policy u is given by

$$u(c,a) = -\frac{\lambda_i}{2} \frac{\partial f}{\partial c}(c,a) = \lambda_i \xi_{ii}(a) c,$$

and therefore,

$$u(x_1) = \eta'(x_1) = \lambda_i \xi_{ii}(a - x_1) \eta(x_1),$$

which can be readily verified.

3.1.3 An alternative proof of Proposition 4. Full discretization scheme I

In Section 3.1.2 we used the fact that the Riccati matrix differential equation (5) can be solved explicitly. This enabled us to verify directly the monotonicity, boundedness and diagonal character of the solution. Let us now employ a different method using a finite difference approximation, which is applicable to the more common cases where we do not possess an explicit analytic representation for the solution.

Proof of Proposition 4. Let us consider $x_1 \in (0, \infty)$. We take $N \in I\!N$, $h = x_1/N$ and $t_i = ih$, for all $i = 0, 1, \cdots N$. The integration can be continued for $i > N$. Now, we approximate $\xi^m(t_i)$ by the matrix $\xi^m_{i,h}$, defined by the explicit scheme

$$\xi^m_{0,h} = 0,$$

and, for $i \geq 1$,

$$\frac{\xi_{i,h}^m - \xi_{i-1,h}^m}{h} = -I^m + \xi_{i-1,h}^m \Lambda^m \xi_{i-1,h}^m. \tag{15}$$

We point out that we could have taken whatever convergent scheme. In Section 3.1.4 we show another possible scheme.

It is easy to prove, by induction, that $\{\xi_{i,h}^m\}_{i=0}^N$ is a sequence of diagonal matrices. Therefore, $\xi^m(x_1) = \lim_{N\to\infty} \xi_{N,x_1/N}^m$ is a diagonal matrix and then P^m has the diagonal form indicated in Proposition 4.

Now, proving that $P^m(x_1) \leq 0$ is equivalent, even for the general non diagonal case, to proving that $\xi^m(x_1) \leq 0$, since, given an arbitrary function $u = \sum_{k=1}^m u_k w_k \in V^m$, we have that

$$(P^m(x_1)u, u) = \left(\sum_{i,j=1}^m u_i \xi_{ij}^m(x_1) w_j, \sum_{k=1}^m u_k w_k \right) = \sum_{i,j=1}^m u_i \xi_{ij}^m(x_1) u_j.$$

Therefore, it suffices to prove the monotonicity properties for $\xi^m(x_1)$. We need the following lemma:

Lemma 3 *There exists a unique diagonal negative definite matrix ξ_∞^m satisfying the matrix equation*

$$-\xi_\infty^m \Lambda^m \xi_\infty^m + I^m = 0. \tag{16}$$

If h satisfies

$$h < \frac{1}{2\rho((\Lambda^m)^{1/2})}, \tag{17}$$

the following monotonicity properties hold for $i < j$:

$$0 \geq \xi_{i,h}^m \geq \xi_{j,h}^m \geq \xi_\infty^m, \tag{18}$$

$$0 \leq \xi_{i,h}^m \Lambda^m \xi_{i,h}^m \leq \xi_{j,h}^m \Lambda^m \xi_{j,h}^m \leq \xi_\infty^m \Lambda^m \xi_\infty^m, \tag{19}$$

$$0 \geq \xi_{i,h}^m + h\xi_{i,h}^m \Lambda^m \xi_{i,h}^m \geq \xi_{j,h}^m + h\xi_{j,h}^m \Lambda^m \xi_{j,h}^m \geq \xi_\infty^m + h\xi_\infty^m \Lambda^m \xi_\infty^m. \tag{20}$$

Furthermore,

$$\lim_{i\to\infty} \xi_{i,h}^m = \xi_\infty^m.$$

Proof. It is clear that ξ_∞^m must satisfy $(\xi_\infty^m)^2 = (\Lambda^m)^{-1}$ and therefore, $\xi_\infty^m = -(\Lambda^m)^{-1/2}$.

Suppose by induction that these properties are satisfied till i. By (15), (16) and (19) we get $\xi_{i,h}^m \geq \xi_{i+1,h}^m$. We have also from (16)

$$\xi_{i+1,h}^m - \xi_\infty^m = \xi_{i,h}^m - \xi_\infty^m + h(\xi_{i,h}^m \Lambda^m \xi_{i,h}^m - \xi_\infty^m \Lambda^m \xi_\infty^m)$$

and so, thanks to (20), $\xi^m_{i+1,h} \geq \xi^m_\infty$. To prove the second monotonicity property let us define the real valued functions $f_i(x) = \lambda_i x^2$, where λ_i is the (i,i) element of the matrix Λ^m. These functions satisfy

$$f_i'(x) = 2\lambda_i x \ (< 0 \text{ if } x < 0)$$

and $f_i(0) = 0$, which implies (by the diagonal character of the matrices involved) the monotonicity property

$$\xi^m_{i,h}\Lambda^m\xi^m_{i,h} \leq \xi^m_{i+1,h}\Lambda^m\xi^m_{i+1,h} \leq \xi^m_\infty\Lambda^m\xi^m_\infty.$$

To obtain the monotonicity property for the sequence $\{\xi^m_{i,h} + h\xi^m_{i,h}\Lambda^m\xi^m_{i,h}\}_{i=0}^\infty$, we define the real valued function $g_i(x) = x + h\lambda_i x^2$, which satisfies $g_i'(x) = 1 + 2h\lambda_i x$ and $g_i(0) = 0$. Now, we need $1 + 2h\lambda_i x \geq 0$ for all $x \in [(\xi^m_\infty)_{ii}, 0]$ and for all $i = 1, \cdots, m$, which is true if

$$h < \frac{1}{2\rho((\Lambda^m)^{1/2})}.$$

By the same reasoning the same properties are true for $i = 1$, and the proof is completed by induction.

By the monotonicity and boundedness of the sequence $\{\xi^m_{i,h}\}_{i=0}^\infty$, it must have a limit which, by its definition must be ξ^m_∞. $\square$

End of the proof of Proposition 4. As we showed above, it suffices to prove the monotonicity property for $\xi^m(x_1)$. We have that

$$\xi^m(x_1) = \lim_{N\to\infty} \xi^m_{N,x_1/N}.$$

Then, $\xi^m(x_1)$ is a diagonal matrix, for $x_1 \geq 0$, satisfying $0 \geq \xi^m(x_1) \geq \xi^m_\infty$. Furthermore,

$$\frac{d\xi^m}{dx_1}(x_1) = \lim_{N\to\infty} \frac{\xi^m_{N,x_1/N} - \xi^m_{N-1,x_1/N}}{h} \leq 0,$$

which implies that, if $0 \leq t_1 \leq t_2$, then

$$0 \geq \xi^m(t_1) \geq \xi^m(t_2) \geq \xi^m_\infty. \quad \square$$

3.1.4 Full discretization Scheme II

In the proof of Lemma 3 we have obtained the stability condition

$$h < \frac{1}{2\rho((\Lambda^m)^{1/2})}.$$

This has not been a problem for our purposes, but, from a computational point of view, it can be interesting to obtain unconditionally stable schemes. An example of this type of schemes is the following one (exactly equivalent to a block Gauss factorization as shown in [7]):

$$\xi^m_{0,h} = 0,$$

and, for $i \geq 1$,

$$\frac{\xi^m_{i,h} - \xi^m_{i-1,h}}{h} = -I^m + \xi^m_{i-1,h}\Lambda^m(I - h\Lambda^m\xi^m_{i-1,h})^{-1}\xi^m_{i-1,h}, \tag{21}$$

(we still use the notation $\xi^m_{i,h}$ for the solution of this new scheme). The proof is given in Lemma 4.

Lemma 4 *There exists a unique diagonal matrix* $\xi^m_{\infty,h} \leq 0$ *satisfying the matrix equation*

$$-\xi^m_{\infty,h}\Lambda^m(I - h\Lambda^m\xi^m_{\infty,h})^{-1}\xi^m_{\infty,h} + I^m = 0. \tag{22}$$

The following monotonicity properties are satisfied for $i < j$*:*

$$0 \geq \xi^m_{i,h} \geq \xi^m_{j,h} \geq \xi^m_{\infty,h}, \tag{23}$$

$$\begin{aligned} 0 \quad &\leq \quad \xi^m_{i,h}\Lambda^m(I - h\Lambda^m\xi^m_{i,h})^{-1}\xi^m_{i,h} \quad \leq \quad \xi^m_{j,h}\Lambda^m(I - h\Lambda^m\xi^m_{j,h})^{-1}\xi^m_{j,h} \\ &\leq \quad \xi^m_{\infty,h}\Lambda^m(I - h\Lambda^m\xi^m_{\infty,h})^{-1}\xi^m_{\infty,h}, \end{aligned} \tag{24}$$

and

$$\begin{aligned} 0 \quad &\geq \quad \xi^m_{i,h} + h\xi^m_{i,h}\Lambda^m(I - h\Lambda^m\xi^m_{i,h})^{-1}\xi^m_{i,h} \\ &\geq \quad \xi^m_{j,h} + h\xi^m_{j,h}\Lambda^m(I - h\Lambda^m\xi^m_{j,h})^{-1}\xi^m_{j,h} \\ &\geq \quad \xi^m_{\infty,h} + h\xi^m_{\infty,h}\Lambda^m(I - h\Lambda^m\xi^m_{\infty,h})^{-1}\xi^m_{\infty,h}. \end{aligned} \tag{25}$$

Furthermore,

$$\lim_{i\to\infty} \xi^m_{i,h} = \xi^m_{\infty,h}.$$

Proof. It suffices to define $\xi^m_{\infty,h}$ as the diagonal matrix with the element (i,i) solution of

$$\begin{cases} \dfrac{\lambda_i x^2}{1 - h\lambda_i x} = 1 \\ x \leq 0, \end{cases}$$

where λ_i is the (i,i) element of matrix Λ^m. This real equation has a unique solution since the real valued function $f(x) = \dfrac{\lambda_i x^2}{1 - h\lambda_i x}$ satisfies

$$f'(x) = \frac{-h\lambda_i^2 x^2 + 2\lambda_i x}{(1 - h\lambda_i x)^2} \; (< 0 \text{ if } x < 0),$$

$f(0) = 0$ and $f(x) \to +\infty$ as $x \to -\infty$. Now the proof by induction is similar to the one of Lemma 3. From the assumption of induction and by (21), (22) and (24) we have $\xi^m_{i,h} \geq \xi^m_{i+1,h}$. From (21), (22) we have

$$\begin{aligned}\xi^m_{i+1,h} - \xi^m_{\infty,h} &= \xi^m_{i,h} - \xi^m_{\infty,h} + h(\xi^m_{i,h}\Lambda^m(I - h\Lambda^m\xi^m_{i,h})^{-1}\xi^m_{i,h} \\ &\quad -\xi^m_{\infty,h}\Lambda^m(I - h\Lambda^m\xi^m_{\infty,h})^{-1}\xi^m_{\infty,h}).\end{aligned}$$

Then by (25), $\xi^m_{i,h} \leq \xi^m_{\infty,h}$. The monotonicity property of the real valued function f, implies

$$\begin{aligned}\xi^m_{i,h}\Lambda^m(I - h\Lambda^m\xi^m_{i,h})^{-1}\xi^m_{i,h} &\leq \xi^m_{i+1,h}\Lambda^m(I - h\Lambda^m\xi^m_{i+1,h})^{-1}\xi^m_{i+1,h} \\ &\leq \xi^m_{\infty,h}\Lambda^m(I - h\Lambda^m\xi^m_{\infty,h})^{-1}\xi^m_{\infty,h}.\end{aligned}$$

Similarly, we can obtain the monotonicity property for the sequence $\{\xi^m_{i,h} + h\xi^m_{i,h}\Lambda^m(I - h\Lambda^m\xi^m_{i,h})^{-1}\xi^m_{i,h}\}_i$, by using the real valued function

$$g(x) = x + \frac{h\lambda_i x^2}{1 - h\lambda_i x} = \frac{x}{1 - h\lambda_i x},$$

which satisfies

$$g'(x) = \frac{1}{(1 - h\lambda_i x)^2} \ (> 0),$$

and $g(0) = 0$. Finally, as in Lemma 3, $\{\xi^m_{i,h}\}_{i=0}^{\infty}$ must have a limit which, by definition, must be $\xi^m_{\infty,h}$. $\square$

As previously we can conclude the proof of Proposition 4 using the facts that

$$\xi^m(x_1) = \lim_{N\to\infty} \xi^m_{N,x_1/N},$$

and

$$\lim_{h\to 0} \xi^m_{\infty,h} = \xi^m_{\infty},$$

which is defined by (16).

3.2 Passing to the limit. Existence of solution. Regularity and Monotonicity Results

We can now define the solution of the Riccati equation (4) in a Hilbert-Schmidt framework.

Theorem 1 *Problem (4) has a solution P limit of P^m in $\mathcal{C}([0,\infty) : \widetilde{H}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{H}') \cap \mathcal{C}^1([0,\infty) : \widetilde{V}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{V}')$ as $m \to \infty$. Furthermore P belongs to $\mathcal{C}([0,+\infty) : \mathcal{L}(V^s, V^{s+1})) \cap \mathcal{C}^1([0,+\infty) : \mathcal{L}(V^s, V^s))$ for all $s \in I\!R$.*

Proof. The proof is based on the passing to the limit of P^m and of the differential equation satisfied by P^m as $m \to \infty$. Now,

$$P^m_\infty = \sum_{i=1}^{m} -\frac{1}{\sqrt{\lambda_i}} w_i \otimes w_i.$$

Therefore, if $0 < m_1 \le m_2$, then $0 \ge P^{m_1}_\infty \ge P^{m_2}_\infty$ and $(P^{m_1}_\infty)_{ii} = (P^{m_2}_\infty)_{ii}$ for $i = 1, \cdots m_1$. Further,

$$P_\infty = \sum_{i=1}^{\infty} -\frac{1}{\sqrt{\lambda_i}} w_i \otimes w_i \in \widetilde{H} \widehat{\otimes}_2 H \cap H \widehat{\otimes}_2 \widetilde{H}',$$

since

$$\|P_\infty\|^2_{\widetilde{H}\widehat{\otimes}_2 H} = \|P_\infty\|^2_{H\widehat{\otimes}_2 \widetilde{H}'} = \sum_{i=1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} < +\infty.$$

Then, $P^m_\infty \to P_\infty$ in the topology of $\widetilde{H}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{H}'$. Now, since we are dealing with diagonal matrices, it is easy to prove that, if $0 < m_1 \le m_2$, then $\xi^{m_1}_{ii}(x_1) = \xi^{m_2}_{ii}(x_1)$ for $i = 1, \cdots, m_1$ and $0 \ge P^{m_1}(x_1) \ge P^{m_2}(x_1) \ge P_\infty$, which implies that, if we define $\xi_{ii} = \xi^i_{ii}$, then

$$P^m \overset{m\to\infty}{\longrightarrow} P = \sum_{i=1}^{\infty} \xi_{ii} w_i \otimes w_i \quad \text{in } \mathcal{C}([0,\infty) : \widetilde{H}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2 \widetilde{H}'),$$

since, using (12),

$$\|P^m - P\|^2_{\mathcal{C}([0,\infty):\widetilde{H}\widehat{\otimes}_2 H)} = \|P^m - P\|^2_{\mathcal{C}([0,\infty):H\widehat{\otimes}_2 \widetilde{H}')}$$

$$= \sup_{x_1\in[0,\infty)} \sum_{i=m+1}^{\infty} \frac{|\xi_{ii}(x_1)|^2}{\lambda_i^\delta} \le \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \overset{m\to\infty}{\longrightarrow} 0.$$

Now (12) gives the expected regularity on P because

$$\|P\|^2_{\mathcal{C}([0,\infty):\mathcal{L}(V^s,V^{s+1}))} = \sup_{x_1\in[0,\infty)} \max_i \lambda_i |\xi_{ii}(x_1)|^2 \le 1.$$

Further, if $0 \le t_1 \le t_2$, then

$$0 \ge P(t_1) \ge P(t_2) \ge P_\infty.$$

We also claim that

$$\frac{dP^m}{dx_1} \overset{m\to\infty}{\longrightarrow} \frac{dP}{dx_1} \quad \text{in } \mathcal{C}([0,\infty) : \widetilde{V}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2 \widetilde{V}'),$$

since, from (12),

$$\left\| \frac{dP^m}{dx_1} - \frac{dP}{dx_1} \right\|^2_{\mathcal{C}([0,\infty):\widetilde{V}\widehat{\otimes}_2 H)} = \left\| \frac{dP^m}{dx_1} - \frac{dP}{dx_1} \right\|^2_{\mathcal{C}([0,\infty):H\widehat{\otimes}_2 \widetilde{V}')}$$

$$= \sup_{x_1\in[0,\infty)} \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \left|\frac{d\xi_{ii}}{dx_1}(x_1)\right|^2 = \sum_{i=m+1}^{\infty} \frac{|1-\lambda_i(\xi_{ii}(x_1))^2|^2}{\lambda_i^{1+\delta}}$$

$$\leq \sum_{i=m+1}^{\infty} \frac{1}{\lambda_i^{1+\delta}} \overset{m\to\infty}{\longrightarrow} 0.$$

From (12) again

$$\left\|\frac{dP}{dx_1}\right\|^2_{\mathcal{C}([0,+\infty):\mathcal{L}(V^s,V^s))} = \sup_{x_1\in[0,\infty)} \max_i |1-\lambda_i(\xi_{ii}(x_1))^2|^2 \leq 1.$$

Similarly, it is easy to deduce that

$$I^m \overset{m\to\infty}{\longrightarrow} I \quad \text{and} \quad P^m\Delta_z P^m \overset{m\to\infty}{\longrightarrow} P\Delta_z P \quad \text{in } \mathcal{C}([0,\infty):\widetilde{V}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{V}').$$

Then $P\in\mathcal{C}([0,\infty):\widetilde{H}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{H}')\cap\mathcal{C}^1([0,\infty):\widetilde{V}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{V}')$ satisfies

$$\frac{dP}{dx_1} + P\Delta_z P + I = 0,$$

where each term is considered as an element of $\widetilde{V}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{V}'$. □

Let $P\in\mathcal{C}([0,\infty):\widetilde{H}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{H}')\cap\mathcal{C}^1([0,\infty):\widetilde{V}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{V}')$ be one of the solutions of problem (4). Then, we have the following strict monotonicity properties:

Theorem 2 *For all* $x_1\geq 0$,

$$\frac{dP}{dx_1}(x_1) < 0,$$

and therefore, if $0 < t_1 < t_2$, *then*

$$0 > P(t_1) > P(t_2) > P_\infty.$$

Proof. It suffices to prove the inequality

$$\frac{d\xi_{ii}}{dx_1}(x_1) < 0,$$

for each Fourier coefficient $\xi_{ii}(x_1)$, $i=1,\ldots,\infty$, which we had already proved in Section 3.1.2. □

3.3 Uniqueness of Solution

Theorem 3 *The solution* P *of (4) is unique in* $\mathcal{C}([0,\infty):\widetilde{H}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{H}')\cap\mathcal{C}^1([0,\infty):\widetilde{V}\widehat{\otimes}_2 H\cap H\widehat{\otimes}_2\widetilde{V}')$.

Proof. Let $P \in \mathcal{C}([0,\infty) : \widetilde{H}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{H}') \cap \mathcal{C}^1([0,\infty) : \widetilde{V}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{V}')$ be a solution of (4). Let

$$P(x_1) = \sum_{i,j=1}^{\infty} \xi_{ij}(x_1) w_i \otimes w_j \in \widetilde{H}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{H}'$$

be the Fourier expansion of $P(x_1)$. Then, equation (4) can be written as

$$\sum_{r,l=1}^{\infty} \left(\frac{d\xi_{rl}}{dx_1}(x_1) - \sum_{k=1}^{\infty} \lambda_k \xi_{kl}^m \xi_{rk}^m \right) w_r \otimes w_l = -\sum_{r=1}^{\infty} w_r \otimes w_r,$$

with both terms of the above equality in $(\widetilde{V}\widehat{\otimes}_2 H) \cap (H\widehat{\otimes}_2\widetilde{V}')$. Then, by the uniqueness of the Fourier representation, we have that

$$\frac{d\xi_{rl}}{dx_1}(x_1) - \sum_{k=1}^{\infty} \lambda_k \xi_{kl}^m \xi_{rk}^m = \begin{cases} -1 & \text{if } r = l, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if ξ^m is the matrix given by $\xi^m = (\xi_{ij})_{i,j=1}^m$, then ξ^m is the unique solution of (6) (as we proved in Section 3.1.1), which concludes the proof due to the strong convergence of ξ^m to ξ in $\mathcal{C}([0,\infty) : \widetilde{H}\widehat{\otimes}_2 H \cap H\widehat{\otimes}_2\widetilde{H}')$ proved in Theorem 1. $\square$

4 Elliptic problems associated to the Riccati Equation (4)

In this section we show how the Hilbert-Schmidt operator P solution of the Riccati equation (4) can be used to transform a second order elliptic boundary value problem into one or two uncoupled first order initial value problems.

We first study the case when $y_0 \equiv 0$ and $f \equiv 0$, implying that $r \equiv 0$ (the linear case), and finally we study the general case (the affine case).

4.1 The linear case

4.1.1 Dirichlet Condition on Γ_a

Theorem 4 *Let $s \in I\!R$ arbitrary and $y_a \in V^s$. Then, there exists a unique solution $y \in \mathcal{C}^r([0,a] : V^{s-r})$, for all $r \in I\!N$, of the initial value problem*

$$\begin{cases} P\dfrac{\partial y}{\partial x_1} = -y, \\ y(a) = y_a. \end{cases} \tag{26}$$

It depends continuously on $y_a \in V^s$. Furthermore y has the additional regularity $y \in \mathcal{C}^\infty([0,a) : V^m)$ for any $m \in I\!N$.

Proof. Let

$$y_a = \sum_{i=1}^{\infty} y_{ai} w_i$$

be the Fourier decomposition of y_a (that is, $y_{ai} = \lambda_i^{-s}(y_a, w_i)_{V^s}$). Then, the solution of the problem is given by

$$y(x_1, x_2) = \sum_{i=1}^{\infty} y_i(x_1) w_i(x_2),$$

where each Fourier coefficient $y_i(x_1)$, $i = 1, \cdots, \infty$, satisfies the differential equation

$$\begin{cases} \xi_{ii} \dfrac{dy_i}{dx_1} = -y_i, \\ y_i(a) = y_{ai}. \end{cases}$$

Then,

$$y_i(x_1) = y_{ai} e^{\int_{x_1}^{a} (\xi_{ii}(t))^{-1} dt} = y_{ai} \frac{\sinh(\sqrt{\lambda_i} x_1)}{\sinh(\sqrt{\lambda_i} a)}.$$

Let us prove that $y \in \mathcal{C}([0,a] : V^s)$. The function $f(x) = \sinh(\sqrt{\lambda_i} x)$ satisfies $f(0) = 0$ and $f'(x) = \sqrt{\lambda_i} \cosh(\sqrt{\lambda_i} x) > 0$ for all $x \in \mathbb{R}$, which implies that

$$\sum_{i=1}^{\infty} \lambda_i^s |y_i(x_1)|^2 \leq \sum_{i=1}^{\infty} \lambda_i^s |y_{ai}|^2 < \infty \tag{27}$$

and therefore $y(x_1, \cdot) \in V^s$ for all $x_1 \in [0,a]$. Further, since $y_i \in \mathcal{C}^\infty([0,a])$, $i = 1, \cdots, \infty$, and thanks to the uniform convergence of the series, we have that $y \in \mathcal{C}([0,a] : V^s)$, since

$$\|y - \sum_{i=1}^{m} y_i w_i\|_{\mathcal{C}([0,a]:V^s)} = \sup_{x_1 \in [0,a]} \sum_{i=m+1}^{\infty} \lambda_i^s |y_i(x_1)|^2 \leq \sum_{i=m+1}^{\infty} \lambda_i^s |y_{ai}|^2 \stackrel{m\to\infty}{\longrightarrow} 0.$$

Let us prove now that $y \in \mathcal{C}^1([0,a] : V^{s-1})$. We have

$$\frac{dy_i}{dx_1}(x_1) = \sqrt{\lambda_i} y_{ai} \frac{e^{\sqrt{\lambda_i} x_1} + e^{-\sqrt{\lambda_i} x_1}}{e^{\sqrt{\lambda_i} a} - e^{-\sqrt{\lambda_i} a}}.$$

Further, the function $f(x) = e^{\sqrt{\lambda_i} x} + e^{-\sqrt{\lambda_i} x}$ satisfies $f(0) = 2$ and $f'(x) = \sqrt{\lambda_i}(e^{\sqrt{\lambda_i} x} - e^{-\sqrt{\lambda_i} x}) \geq 0$ for all $x \geq 0$, which implies that, for all $x_1 \in [0,a]$,

$$\sum_{i=1}^{\infty} \lambda_i^{s-1} |\frac{dy_i}{dx_1}(x_1)|^2 \leq \sum_{i=1}^{\infty} \lambda_i^s |y_{ai}|^2 \left(\frac{e^{\sqrt{\lambda_i} a} + e^{-\sqrt{\lambda_i} a}}{e^{\sqrt{\lambda_i} a} - e^{-\sqrt{\lambda_i} a}} \right)^2.$$

Now, the function

$$f(x) = \frac{e^{ax} + e^{-ax}}{e^{ax} - e^{-ax}}$$

satisfies

$$f'(x) = \frac{-4a}{(e^{ax} - e^{-ax})^2} < 0, \quad \forall x \in I\!R.$$

Therefore,

$$\sum_{i=1}^{\infty} \lambda_i^{s-1} |\frac{dy_i}{dx_1}(x_1)|^2 \leq \left(\frac{1}{\tanh(\sqrt{\lambda_1}a)}\right)^2 \sum_{i=1}^{\infty} \lambda_i^s |y_{ai}|^2 < \infty, \tag{28}$$

which implies that $\dfrac{dy}{dx_1}(x_1, \cdot) \in V^{s-1}$, for all $x_1 \in [0, a]$, and therefore $y \in \mathcal{C}^1([0, a] : V^{s-1})$. The continuous dependence on the initial data is a consequence of (27), (28). Finally, taking into account that, for all $r \in I\!N$,

$$\frac{d^r y_i}{dx_1^r}(x_1) = \lambda_i^{r/2} y_{ai} \frac{e^{\sqrt{\lambda_i}x_1} + (-1)^{r+1} e^{-\sqrt{\lambda_i}x_1}}{e^{\sqrt{\lambda_i}a} - e^{-\sqrt{\lambda_i}a}},$$

it is easy to prove that $y \in \mathcal{C}^r([0, a] : V^{s-r})$.

To prove the additional regularity we consider

$$\begin{aligned}
\sum_{i=1}^{\infty} \lambda_i^m |y_i(x_1)|^2 &\leq \sum_{i=1}^{\infty} \lambda_i^s |y_{ai}|^2 \frac{\lambda_i^{m-s} (e^{\sqrt{\lambda_i}(x_1-a)} - e^{-\sqrt{\lambda_i}(x_1+a)})^2}{(1 - e^{-2\sqrt{\lambda_i}a})^2} \\
&\leq \frac{1}{(1 - e^{-2\sqrt{\lambda_1}a})^2} \sum_{i=1}^{\infty} \lambda_i^s |y_{ai}|^2 \lambda_i^{m-s} e^{2\sqrt{\lambda_i}(x_1-a)} < \infty
\end{aligned}$$

for any $x_1 < a$, since, for any $\mu, k > 0$, $\lambda^k e^{-\mu\lambda} \to 0$ as $\lambda \to \infty$. The proof is similar for any derivative. $\square$

Theorem 5 *Given $y_a \in V^s$, the solution $y \in \mathcal{C}^r([0, a] : V^{s-r})$ of (26) is also the unique solution of the elliptic problem*

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \\ y = 0 & \text{on } \Gamma_0, \\ y = y_a & \text{on } \Gamma_a. \end{cases} \tag{29}$$

Proof. The equation of problem (26) is satisfied in $\mathcal{C}^r([0, a] : V^{s-r})$, for all $r \in I\!N$. Then, derivating with respect to the x_1-variable we get

$$\frac{\partial y}{\partial x_1} = -\frac{dP}{dx_1}\frac{\partial y}{\partial x_1} - P\frac{\partial^2 y}{\partial x_1^2},$$

which is an equation with all its terms in $\mathcal{C}([0, a] : V^{s-1})$ (see Theorem 1 and regularity results of Theorem 4). Then, taking into account the equation satisfied by the Hilbert-Schmidt operator P, we have that

$$P\frac{\partial^2 y}{\partial x_1^2} - P\Delta_z P\frac{\partial y}{\partial x_1} = 0.$$

Therefore, we have the following equation in $\mathcal{C}([0,a]:V^{s-1})$:

$$P\left(\frac{\partial^2 y}{\partial x_1^2}+\Delta_z y\right)=0.$$

Hence, since $P(x_1)<0$ for all $x_1>0$, we have that

$$-\Delta y=-\frac{\partial^2 y}{\partial x_1^2}-\Delta_z y=0 \quad \text{in } \mathcal{C}([0,a]:V^{s-2}).$$

Finally, the boundary conditions are obviously satisfied, which concludes the proof. □

Remark 6 The regularity result in Theorem 4 appears as the result of the well known property of regularization of parabolic operators away from the initial condition. The same result in the context of Theorem 5 can be viewed as an interior regularity property for elliptic equations.

Example 2 Let $\Omega=(0,a)\times(0,b)$ (that is, $\mathcal{O}=(0,b)$). Then,

$$\begin{cases} \lambda_n=(\frac{n\pi}{b})^2, \ n\geq 1 \text{ and } n\in I\!N, \\ w_n(x_2)=\sqrt{\frac{2}{b}}\sin(\frac{n\pi}{b}x_2). \end{cases}$$

Therefore, the solution of (29) is

$$y(x_1,x_2)=\sqrt{\frac{2}{b}}\sum_{n=1}^{\infty} y_{an}\frac{e^{\frac{n\pi}{b}x_1}-e^{-\frac{n\pi}{b}x_1}}{e^{\frac{n\pi}{b}a}-e^{-\frac{n\pi}{b}a}}\sin(\frac{n\pi}{b}x_2),$$

which can be approximated by

$$y^m(x_1,x_2)=\sqrt{\frac{2}{b}}\sum_{n=1}^{m} y_{an}\frac{e^{\frac{n\pi}{b}x_1}-e^{-\frac{n\pi}{b}x_1}}{e^{\frac{n\pi}{b}a}-e^{-\frac{n\pi}{b}a}}\sin(\frac{n\pi}{b}x_2),$$

for m large enough.

For instance, if $y_a\equiv 1$, then $y_a\in L^2(0,b)$ and

$$y_{an}=\begin{cases} \frac{2\sqrt{2b}}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

Therefore,

$$y(x_1,x_2)=\sum_{n=1}^{\infty}\frac{4}{(2n-1)\pi}\frac{e^{\frac{(2n-1)\pi}{b}(x_1-a)}-e^{-\frac{(2n-1)\pi}{b}(x_1+a)}}{1-e^{-2\frac{(2n-1)\pi}{b}a}}\sin(\frac{(2n-1)\pi}{b}x_2),$$

Figure 1 (respectively Figure 2) shows the linear interpolation of y^{20} (respectively y^{50}), over a grid of 16 elements for the x_1-coordinate and 26 elements for the x_2-coordinate, when $y_a\equiv 1$, $a=3$ and $b=5$.

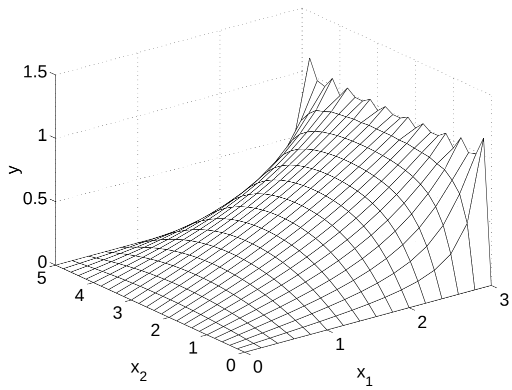

Figure 1: Graph of the function y^{20} of Example 2.

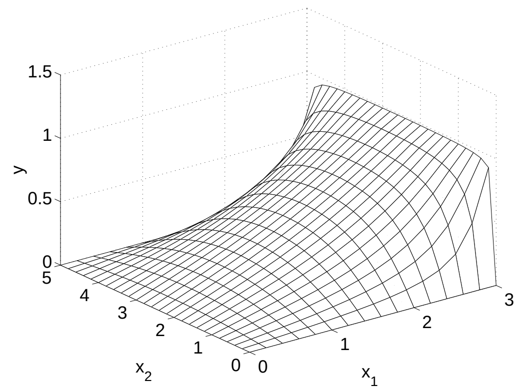

Figure 2: Graph of the function y^{50} of Example 2.

4.1.2 Neumann Condition on Γ_a

In order to use the operator P for problems with a Neumann boundary condition on Γ_a as problem (1) we shall need the following corollary.

Corollary 2 *Let $s \in I\!R$ arbitrary and $y_a \in V^s$. Then, there exists a unique solution $y \in \mathcal{C}^r([0,a] : V^{s+1-r})$, for all $r \in I\!N$, of the initial value problem*

$$\begin{cases} P\dfrac{\partial y}{\partial x_1} = -y, \\ y(a) = -P(a)y_a. \end{cases} \tag{30}$$

It depends continuously on $y_a \in V^s$. Furthermore, y has the additional regularity $y \in \mathcal{C}^\infty([0,a] : V^m)$ for any $m \in I\!N$.

Proof. It suffices to apply Theorem 1 (to show that $-P(a)y_a \in V^{s+1}$) and Theorem 4. □

Theorem 6 *Given $y_a \in V^s$, the solution $y \in \mathcal{C}^r([0,a] : V^{s+1-r})$ of (30) is also the unique solution of the elliptic problem*

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \\ y = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial y}{\partial x_1} = y_a & \text{on } \Gamma_a. \end{cases} \tag{31}$$

Proof. The proof is analogous to the proof given in Theorem 5. □

4.1.3 Transparent Boundary Conditions

The following theorem shows how it is possible to use P to define a transparent boundary condition on $\Gamma_t = \{t\} \times \mathcal{O}$ for $0 < t < a$ in order that the solution of a boundary value problem defined on the subdomain $\Omega_t = (t,a) \times \mathcal{O}$ is exactly the restriction to Ω_t of the solution of (29).

Theorem 7 *Given $y_a \in V^s$, the solution $y \in \mathcal{C}^r([0,a] : V^{s-r})$ of (26) is also the unique solution of the elliptic problem*

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega_t, \\ y = 0 & \text{on } \Sigma, \\ -P\dfrac{\partial y}{\partial x_1} = y & \text{on } \Gamma_t, \\ y = y_a & \text{on } \Gamma_a, \end{cases} \tag{32}$$

where $\Omega_t = (t,a) \times \mathcal{O}$.

Proof. We only need to prove that problem (32) has a unique solution. Now, the variational formulation is

$$\begin{cases} \text{Find } y \in \{\varphi \in H^1(\Omega_t) : \varphi|_\Sigma = 0, \varphi|_{\Gamma_a} = y_a\}, \text{ such that} \\ a(y,\varphi) = 0 \ \forall \ \varphi \in \mathcal{U}_t = \{\varphi \in H^1(\Omega_t) : \varphi|_\Sigma = 0, \varphi|_{\Gamma_a} = 0\}, \end{cases}$$

where

$$a(y,\varphi) = \int_\Omega \nabla y \nabla \varphi dx_1 dz - \int_{\mathcal{O}} P^{-1}(t) y(t) \varphi(t) dz.$$

Now, since $P(t)$ is a negative definite operator, it is easy to prove, by means of the Poincaré inequality, that $a(\cdot,\cdot)$ is a positive definite bilinear form on $\mathcal{U}_t$. This concludes the result by using the Lax-Milgram Theorem (see Section V.3 of [4]). □

Similarly, we have the following result:

Theorem 8 *Given $y_a \in V^s$, the solution $y \in \mathcal{C}^r([0,a] : V^{s+1-r})$ of (30) is also the unique solution of the elliptic problem*

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega_t, \\ y = 0 & \text{on } \Sigma, \\ -P\dfrac{\partial y}{\partial x_1} = y & \text{on } \Gamma_t, \\ \dfrac{\partial y}{\partial x_1} = y_a & \text{on } \Gamma_a. \end{cases}$$

Proof. The proof is analogous to the proof given in Theorem 7. □

Remark 7 Theorems 5 and 7 (respectively, 6 and 8) imply that the solution $y \in \mathcal{C}^r([0,a] : V^{s-r})$ of (29) (respectively, (31)) is also solution of the elliptic problem

$$\begin{cases} -\Delta y = 0 & \text{in } \Omega_t^*, \\ y = 0 & \text{on } \Sigma, \\ y = 0 & \text{on } \Gamma_0, \\ -P\dfrac{\partial y}{\partial x_1} = y & \text{on } \Gamma_t, \end{cases}$$

where $\Omega_t^* = (0,t) \times \mathcal{O}$. We point out that this problem is not well-posed, since it does not have a unique solution (for instance, $y \equiv 0$, the solution of (29) and the solution of (31) are solutions of that problem). However, we can deduce that, given an arbitrary function $h \in V^s$, we have that $P(t)h = \gamma|_{\Gamma_t}$, where $\gamma \in \mathcal{C}([0,t] : V^{s+1})$ is the unique solution of

$$\begin{cases} -\Delta \gamma = 0 & \text{in } \Omega_t^*, \\ \gamma = 0 & \text{on } \Sigma, \\ \gamma = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial \gamma}{\partial x_1} = -h & \text{on } \Gamma_t, \end{cases}$$

which was the initial definition given in [10], [7] for operator P, before deducing the associated Riccati equation.

4.1.4 Associated Optimal Control Problems

The solution of (31) is given by

$$y(x_1, x_2) = \sum_{i=1}^{\infty} y_i(x_1) w_i(x_2),$$

with

$$y_i(x_1) = -\frac{y_{ai}}{\sqrt{\lambda_i}} \frac{\sinh(\sqrt{\lambda_i} x_1)}{\cosh(\sqrt{\lambda_i} a)}.$$

Thus, if $\overline{y}_i(x_1) = y_i(a - x_1)$ and $\eta(x_1) = \dfrac{d\overline{y_i}}{dx_1}(x_1)$, then

$$\eta(x_1) = y_{ai} \frac{\cosh\left(\sqrt{\lambda_i}(a - x_1)\right)}{\cosh(\sqrt{\lambda_i} a)}$$

is the solution of the Euler equation (14) and, therefore, the optimal trajectory to the minimization problem (13), with $c = y_{ai}$.

Now, if we consider the whole Fourier expansion of the solution we have that $\dfrac{\partial y}{\partial x}$ is the optimal trajectory of suitable optimal control problems, depending on the regularity of y_a. Let us consider a couple of examples:

Let us suppose that $y_a \in V^{-1/2} = H_{00}^{1/2}(\mathcal{O})'$. Then, considering the whole Fourier expansion of the solution we have that $\dfrac{\partial y}{\partial x_1}$ is the optimal trajectory of

$$\int_0^a \left\| \frac{\partial h}{\partial x_1} \right\|_{H^{-1}(\mathcal{O})}^2 dx_1 + \int_0^a \| h \|_{L^2(\mathcal{O})}^2 \, dx_1 \quad \forall\, h \in X_{y_a},$$

where

$$X_{y_a} = \{h \in L^2(0,a;L^2(\mathcal{O})) \cap H^1(0,a;H^{-1}(\mathcal{O})) : h(a) = y_a\}$$

(we point out that $X_{y_a} \subset \mathcal{C}([0,a] : H_{00}^{1/2}(\mathcal{O})'))$.

Similarly, if we suppose now that $y_a \in V^{1/2} = H_{00}^{1/2}(\mathcal{O})$, then, since $\dfrac{\partial y_i}{\partial x}$ is also the optimal trajectory of

$$\int_0^a h_i'^2 + \lambda_i h_i^2 \quad \forall\, h_i \in \mathcal{U}_{y_{ai}}^* = \{\varphi \in H^1(0,a) : \varphi(a) = y_{a_i}\},$$

considering the whole Fourier expansion of the solution, we have that $\dfrac{\partial y}{\partial x}$ is the optimal trajectory of

$$\int_0^a \left\| \frac{\partial h}{\partial x_1} \right\|_{L^2(\mathcal{O})}^2 dx_1 + \int_0^a \| \nabla_z h \|_{L^2(\mathcal{O})}^2 \, dx_1 \quad \forall\, h \in X_{y_a},$$

where

$$X_{y_a} = \{h \in L^2(0,a;H_0^1(\mathcal{O})) \cap H^1(0,a;L^2(\mathcal{O})) : h(a) = y_a\}$$

(we point out that $X_{y_a} \subset \mathcal{C}([0,a] : H_{00}^{1/2}(\mathcal{O})))$.

Remark 8 Let y^1 be the solution of (26) (and (29)) and y^2 be the solution of (30) (and (31)). Then

$$y^2(a,\cdot) = -P(a)y^1(a,\cdot).$$

Now multiplying (26) by $-P(a)$ and using the fact that $P(a)$ and $P(x_1)$ commute, we obtain that this relation is true for any $x_1 \in (0,a)$

$$y^2(x_1,\cdot) = -P(a)y^1(x_1,\cdot).$$

Therefore we have that $-P(a)\dfrac{dy^1}{dx_1}$ $(= \dfrac{dy^2}{dx_1})$ is the optimal trajectory of the optimal control problems specified above, depending on the regularity of y_a.

4.2 The affine case

We now consider the Poisson problem with non zero condition on Γ_0 or right hand side f.

4.2.1 Function r

Theorem 9 *Let $s \in \mathbb{R}$ arbitrary and $y_0 \in V^s$. Then, there exists a unique solution $r \in \mathcal{C}^k([0,a] : V^{s-k}) \cap L^2(0,a : V^{s+1/2}) \cap H^1(0,a : V^{s-1/2})$, for all $k \in \mathbb{N}$, of the initial value problem*

$$\begin{cases} \dfrac{dr}{dx_1} + P\Delta_z r = 0, \\ r(0) = y_0. \end{cases}$$

Furthermore, r has the additional regularity $r \in \mathcal{C}^\infty((0,a] : V^m)$, for any $m \in \mathbb{N}$.

Proof. Let

$$y_0 = \sum_{i=1}^{\infty} y_{0i} w_i$$

be the Fourier decomposition of y_0. Then, the solution of the problem is given by

$$r(x_1, x_2) = \sum_{i=1}^{\infty} r_i(x_1) w_i(x_2),$$

where each Fourier coefficient $r_i(x_1)$, $i = 1, \cdots, \infty$, satisfies the differential equation

$$\begin{cases} \dfrac{dr_i}{dx_1} = \lambda_i \xi_{ii} r_i, \\ r_i(0) = y_{0i}. \end{cases}$$

Then,

$$r_i(x_1) = y_{0i} e^{\int_0^{x_1} \lambda_i \xi_{ii}(t) dt} = \frac{y_{0i}}{\cosh\left(\sqrt{\lambda_i} x_1\right)}.$$

Now, for all $x_1 \in [0,a]$, we have that

$$\sum_{i=1}^{\infty} \lambda_i^s |r_i(x_1)|^2 \leq \sum_{i=1}^{\infty} \lambda_i^s |y_{0i}|^2 < \infty,$$

which implies that $r(x_1) \in V^s$. Hence, since $r_i \in \mathcal{C}^\infty([0,a])$, $i = 1, \cdots, \infty$, and thanks to the uniform convergence of the preceding series, we have that $r \in \mathcal{C}([0,a] : V^s)$. To obtain the $L^2(0,a : V^{s+1/2})$ regularity consider

$$\begin{aligned} \int_0^a \sum_{i=1}^{\infty} \lambda_i^{s+1/2} |r_i(x_1)|^2 dx_1 &\leq 4 \sum_{i=1}^{\infty} \lambda_i^s |y_{0i}|^2 \int_0^a \frac{\sqrt{\lambda_i}}{e^{2\sqrt{\lambda_i} x_1}} dx_1 \\ &= 2 \sum_{i=1}^{\infty} (1 - e^{-2\sqrt{\lambda_i} a}) \lambda_i^s |y_{0i}|^2 < \infty. \end{aligned}$$

Let us prove now that $r \in \mathcal{C}^1([0,a]:V^{s-1})$. We have

$$\frac{dr_i}{dx_1}(x_1) = -\sqrt{\lambda_i} y_{0i} \frac{\sinh(\sqrt{\lambda_i}x_1)}{(\cosh(\sqrt{\lambda_i}x_1))^2},$$

which implies that, for all $x_1 \in [0,a]$,

$$\sum_{i=1}^{\infty} \lambda_i^{s-1} \left|\frac{dr_i}{dx_1}(x_1)\right|^2 \leq 4 \sum_{i=1}^{\infty} \lambda_i^s |y_{0i}|^2 < \infty.$$

Thus, $\dfrac{dr}{dx_1}(x_1,\cdot) \in V^{s-1}$, for all $x_1 \in [0,a]$, and therefore $r \in \mathcal{C}^1([0,a]:V^{s-1})$.

Further, taking into account that, for all $k \in I\!N$, there exists a constant $c(k) > 0$ such that

$$\left|\frac{d^k r_i}{dx_1^k}(x_1)\right| = c(k)|\lambda_i^{k/2} y_{0i}|,$$

it is easy to prove that $r \in \mathcal{C}^k([0,a]:V^{s-k})$.

To obtain the $H^1(0,a:V^{s-1/2})$ regularity consider

$$\begin{aligned} \int_0^a \sum_{i=1}^{\infty} \lambda_i^{s-1/2} \left|\frac{dr_i}{dx_1}(x_1)\right|^2 dx_1 &\leq 4 \sum_{i=1}^{\infty} \lambda_i^s |y_{0i}|^2 \int_0^a \frac{\sqrt{\lambda_i}}{e^{2\sqrt{\lambda_i}x_1}} dx_1 \\ &= 2 \sum_{i=1}^{\infty} (1 - e^{-2\sqrt{\lambda_i}a}) \lambda_i^s |y_{0i}|^2 < \infty. \end{aligned}$$

To prove the additional regularity we consider

$$\sum_{i=1}^{\infty} \lambda_i^{s-1/2} |r_i(x_1)|^2 \leq \sum_{i=1}^{\infty} \lambda_i^{s-1/2} |y_{0i}|^2 \lambda_i^{m-s} e^{-2\sqrt{\lambda_i}x_1} < \infty$$

for any $x_1 > 0$. $\square$

Theorem 10 *Let $s \in I\!R$ arbitrary and $f \in \mathcal{C}([0,a]:V^s)$ (respectively, $f \in L^2(0,a:V^s)$). Then, there exists a unique solution $r \in \mathcal{C}([0,a]:V^{s+1}) \cap \mathcal{C}^1([0,a]:V^s)$ (respectively, $r \in \mathcal{C}([0,a]:V^{s+1})$), of the initial value problem*

$$\begin{cases} \dfrac{dr}{dx_1} + P\Delta_z r = -Pf, \\ r(0) = 0, \end{cases}$$

and r has also the regularity $r \in L^2(0,a;V^{s+3/2}) \cap H^1(0,a:V^{s+1/2})$ (respectively the same result).

Proof. Let

$$f(x_1,\cdot)=\sum_{i=1}^{\infty} f_i(x_1)w_i$$

be the Fourier decomposition of $f(x_1,\cdot)$, for all $x_1\in[0,a]$ (respectively, a.e. in $(0,a)$), with $f_i\in\mathcal{C}([0,a])$ (respectively, $f_i\in L^2(0,a)$). Then, the solution of the problem is given by

$$r(x_1,x_2)=\sum_{i=1}^{\infty} r_i(x_1)w_i(x_2),$$

where each Fourier coefficient $r_i(x_1)$, $i=1,\cdots,\infty$, satisfies the differential equation

$$\left\{\begin{array}{l}\dfrac{dr_i}{dx_1}=\xi_{ii}\,(\lambda_i r_i-f_i),\\ r_i(0)=0.\end{array}\right. \tag{33}$$

Then (see Appendix A.2),

$$r_i(x_1)=\frac{1}{\sqrt{\lambda_i}}\frac{\displaystyle\int_0^{x_1}\sinh{(\sqrt{\lambda_i}t)}f_i(t)dt}{\cosh{(\sqrt{\lambda_i}x_1)}}. \tag{34}$$

Now, for all $x_1\in[0,a]$ (respectively, a.e. in $(0,a)$), we have that

$$|r_i(x_1)|\le\frac{1}{\sqrt{\lambda_i}}\int_0^{x_1}|f_i(t)|dt\le\sqrt{\frac{a}{\lambda_i}}\|f_i\|_{L^2(0,a)},$$

and, therefore,

$$\sum_{i=1}^{\infty}\lambda_i^{s+1}|r_i(x_1)|^2\le a\sum_{i=1}^{\infty}\lambda_i^{s}\|f_i\|^2_{L^2(0,a)}=a\|f\|^2_{L^2(0,a:V^s)}<\infty,$$

which implies that $r(x_1)\in V^{s+1}$. Hence, since $r_i\in\mathcal{C}^\infty([0,a])$ (because $f_i\in L^1(0,a)$), $i=1,\cdots,\infty$, and thanks to the uniform convergence of the series we have that $r\in\mathcal{C}([0,a]:V^s)$.

By the Cauchy-Schwartz inequality we have

$$\begin{aligned}\int_0^{x_1}\left(e^{\sqrt{\lambda_i}t}-e^{-\sqrt{\lambda_i}t}\right)f_i(t)dt &\le \left(\int_0^{x_1}e^{2\sqrt{\lambda_i}t}dt\right)^{1/2}\|f_i\|_{L^2(0,a)}\\ &\le \frac{1}{\sqrt{2}\lambda_i^{1/4}}e^{\sqrt{\lambda_i}x_1}\|f_i\|_{L^2(0,a)}.\end{aligned}$$

Then

$$\begin{aligned}\int_0^a\sum_{i=1}^{\infty}\lambda_i^{s+3/2}|r_i(x_1)|^2dx_1 &\le \int_0^a\left(\sum_{i=1}^{\infty}\lambda_i^{s+1/2}\frac{e^{2\sqrt{\lambda_i}x_1}\|f_i\|^2_{L^2(0,a)}}{2\lambda_i^{1/2}(e^{\sqrt{\lambda_i}x_1}+e^{-\sqrt{\lambda_i}x_1})^2}\right)dx_1\\ &\le \int_0^a\left(\sum_{i=1}^{\infty}\frac{\lambda_i^s}{2}\|f_i\|^2_{L^2(0,a)}\right)dx_1=\frac{a}{2}\|f\|^2_{L^2(0,a:V^s)}.\end{aligned}$$

From that inequality we get $r \in L^2(0,a;V^{s+3/2})$. Finally, since $\xi_{ii} \in \mathcal{C}^\infty([0,a])$ is uniformly bounded, from (33) and Theorem 1, we deduce that $r \in \mathcal{C}^1([0,a]:V^s) \cap H^1(0,a:V^{s+1/2})$ (respectively, $r \in H^1(0,a:V^{s+1/2})$). $\square$

Remark 9 More general results than those of Theorem 10 can be obtained by taking function f in other spaces (as, for instance, $\mathcal{C}^k([0,a]:V^s)$ or $H^{-1}(\Omega)$).

Corollary 3 *Let $j,k \in \mathbb{R}$ arbitrary, $y_0 \in V^j$ and $f \in \mathcal{C}([0,a]:V^k)$ (respectively, $f \in L^2(0,a:V^k)$). Then, if $p = \min\{j,k+1\}$, there exists a unique solution $r \in \mathcal{C}([0,a]:V^p) \cap \mathcal{C}^1([0,a]:V^{p-1}) \cap L^2(0,a:V^{p+1/2}) \cap H^1(0,a:V^{p-1/2})$ (respectively, $r \in \mathcal{C}([0,a]:V^p) \cap H^1(0,a:V^{p-1/2})$), of the initial value problem*

$$\left\{ \begin{array}{l} \dfrac{dr}{dx_1} + P\Delta_z r = -Pf, \\ r(0) = y_0. \end{array} \right. \tag{35}$$

Furthermore, each Fourier coefficient r_i, $i = 1,\cdots,\infty$, is given by

$$r_i(x_1) = \frac{1}{\sqrt{\lambda_i}\cosh(\sqrt{\lambda_i}x_1)}\left(\int_0^{x_1}\cosh(\sqrt{\lambda_i}t)f_i(t)dt + y_{0i}\right).$$

4.2.2 Dirichlet Condition on Γ_a

In the following Lemma 5 and Corollary 4 we give existence and uniqueness results for the corresponding solution y in a suitable functional space which enables us to prove that y is solution of a Poisson equation. Therefore we do not try to obtain the best regularity results, since they can be obtained later as a result of the well-known regularity results for the Poisson equation.

Lemma 5 *Let $j,k \in \mathbb{R}$ arbitrary, $y_0 \in V^j$ and $f \in \mathcal{C}([0,a]:V^k)$ (respectively, $f \in L^2(0,a:V^k)$). Then, if $p = \min\{j,k+1\}$, there exists a unique solution $y \in \mathcal{C}([0,a]:V^{p-1}) \cap \mathcal{C}^1((0,a]:V^{p-2}) \cap \mathcal{C}^2((0,a]:V^{p-3})$ (respectively, $y \in \mathcal{C}([0,a]:V^{p-1}) \cap H^1(0,a:V^{p-2}) \cap H^2(0,a:V^{p-3})$), of the initial value problem*

$$\left\{ \begin{array}{l} P\dfrac{\partial y}{\partial x_1} = -y + r, \\ y(a) = 0, \end{array} \right.$$

where $r \in \mathcal{C}([0,a]:V^p) \cap \mathcal{C}^1[0,a]:V^{p-1}) \cap L^2(0,a:V^{p+1/2}) \cap H^1(0,a:V^{p-1/2})$ (respectively, $r \in \mathcal{C}([0,a]:V^p) \cap H^1(0,a:V^{p-1/2})$) is the solution of the initial value problem (35).

Proof. The solution of the problem is given by

$$y(x_1,x_2) = \sum_{i=1}^{\infty} y_i(x_1)w_i(x_2),$$

where each Fourier coefficient $y_i(x_1)$, $i = 1, \cdots, \infty$, satisfies the differential equation

$$\begin{cases} \xi_{ii} \dfrac{dy_i}{dx_1} = -y_i + r_i, \\ y_i(a) = 0. \end{cases} \tag{36}$$

Then (see Appendix A.3),

$$y_i(x_1) = \sqrt{\lambda_i} \sinh(\sqrt{\lambda_i} x_1) \int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt. \tag{37}$$

Let us prove that $y \in \mathcal{C}([0,a] : V^{p-1})$. It is easy to prove that the function $f(x) = \frac{\cosh(\sqrt{\lambda_i} x)}{(\sinh(\sqrt{\lambda_i} x))^2}$ satisfies $f(x) > 0$ and $f'(x) < 0$ for all $x > 0$, which implies that

$$\begin{aligned} \sum_{i=1}^{\infty} \lambda_i^{p-1} |y_i(x_1)|^2 &\leq a \sum_{i=1}^{\infty} \lambda_i^p \frac{1}{\left(\tanh(\sqrt{\lambda_i} x_1)\right)^2} \|r_i\|^2_{L^2(0,a)} \\ &\leq \frac{a}{\left(\tanh(\sqrt{\lambda_i} x_1)\right)^2} \sum_{i=1}^{\infty} \lambda_i^p \|r_i\|^2_{L^2(0,a)} \\ &= \frac{a}{\left(\tanh(\sqrt{\lambda_i} x_1)\right)^2} \|r\|^2_{L^2(0,a:V^p)} < \infty \end{aligned} \tag{38}$$

and therefore $y(x_1, \cdot) \in V^{p-1}$ for all $x_1 \in (0,a]$. Further, since $y_i \in \mathcal{C}((0,a])$, $i = 1, \cdots, \infty$, we have that $y \in \mathcal{C}((0,a] : V^{p-1})$. Now, since $r_i \in \mathcal{C}([0,a])$, it is easy to prove that, if $r_i(0) = y_{0i} \neq 0$, then

$$\lim_{x_1 \to 0} \left| \int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt \right| = \infty. \tag{39}$$

Therefore, if $y_{0i} = 0$ and (39) does not hold, then (obviously)

$$\lim_{x_1 \to 0} y_i(x_1) = 0.$$

Otherwise,

$$\begin{aligned} \lim_{x_1 \to 0} y_i(x_1) &= \lim_{x_1 \to 0} \frac{\sqrt{\lambda_i} \displaystyle\int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt}{\dfrac{1}{\sinh(\sqrt{\lambda_i} x_1)}} \\ &= \lim_{x_1 \to 0} \frac{\sqrt{\lambda_i} r_i(x_1) \dfrac{\cosh(\sqrt{\lambda_i} x_1)}{\left(\sinh(\sqrt{\lambda_i} x_1)\right)^2}}{\sqrt{\lambda_i} \dfrac{\cosh(\sqrt{\lambda_i} x_1)}{\left(\sinh(\sqrt{\lambda_i} x_1)\right)^2}} = r_i(0) = y_{0i}. \end{aligned}$$

This implies that $y_i \in \mathcal{C}([0,a])$ for all $i = 1, \cdots, \infty$. Now, to see that $y \in \mathcal{C}([0,a] : V^{p-1})$, we notice that

$$\|y - \sum_{i=1}^{m} y_i w_i\|_{\mathcal{C}([0,a]:V^{p-1})} = \sup_{x_1 \in [0,a]} \sum_{i=m+1}^{\infty} \lambda_i^{p-1} |y_i(x_1)|^2 \to 0 \quad \text{as} \quad m \to \infty,$$

since in other case, there exists $\varepsilon > 0$ and $\{x_1^m\}_{m \in I\!N} \subset [0,a]$ such that

$$\sum_{i=m+1}^{\infty} \lambda_i^{p-1} |y_i(x_1^m)|^2 > \varepsilon \quad \forall \ m \in I\!N$$

and then, by (38), $x_1^m \to 0$ as $m \to \infty$, which implies that $y_0 \notin V^{p-1}$, in contradiction with the assumptions.

Let us prove now that $y \in \mathcal{C}^1((0,a] : V^{p-2}) \cap \mathcal{C}^2((0,a] : V^{p-3})$. We have

$$\begin{aligned} \frac{dy_i}{dx_1}(x_1) &= \lambda_i \cosh(\sqrt{\lambda_i} x_1) \int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt \\ &\quad -\sqrt{\lambda_i} r_i(x_1) \coth(\sqrt{\lambda_i} x_1). \end{aligned}$$

and

$$\begin{aligned} \frac{d^2 y_i}{dx_1^2}(x_1) &= \lambda_i^{3/2} \sinh(\sqrt{\lambda_i} x_1) \int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt \\ &\quad -\sqrt{\lambda_i} \frac{dr_i}{dx_1}(x_1) \coth(\sqrt{\lambda_i} x_1) - \lambda_i r_i(x_1). \end{aligned}$$

Then, in a way similar to that followed above, it is easy to prove that $\frac{dy_i}{dx_1}(x_1) \in V^{p-2}$ and $\frac{d^2 y_i}{dx_1^2}(x_1) \in V^{p-3}$ for all $x_1 \in (0,a]$ (respectively, a.e. in $(0,a]$). Further, since $y_i \in \mathcal{C}^2((0,a])$ (respectively, $y_i \in H^2(0,a)$), we have that $\frac{dy_i}{dx_1} \in \mathcal{C}((0,a] : V^{p-2})$ and $\frac{d^2 y_i}{dx_1^2} \in \mathcal{C}((0,a] : V^{p-3})$ (respectively, $\frac{dy_i}{dx_1} \in L^2(0,a : V^{p-2})$ and $\frac{d^2 y_i}{dx_1^2} \in L^2(0,a : V^{p-3})$), $i = 1, \cdots, \infty$. $\square$

Corollary 4 *Let $j,k,q \in I\!R$ arbitrary, $y_0 \in V^j$, $f \in \mathcal{C}([0,a] : V^k)$ (respectively, $f \in L^2(0,a : V^k)$) and $y_a \in V^q$. Then, if $p = \min\{j, k+1, q+1\}$, there exists a unique solution $y \in \mathcal{C}([0,a] : V^{p-1}) \cap \mathcal{C}^1((0,a] : V^{p-2}) \cap \mathcal{C}^2((0,a] : V^{p-3})$ (respectively, $y \in \mathcal{C}([0,a] : V^{p-1}) \cap H^1(0,a : V^{p-2}) \cap H^2(0,a : V^{p-3})$), of the initial value problem*

$$\left\{ \begin{array}{l} P \dfrac{\partial y}{\partial x_1} = -y + r, \\ y(a) = y_a, \end{array} \right. \tag{40}$$

where $r \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1[0,a] : V^{p-1}) \cap L^2(0,a : V^{p+1/2}) \cap H^1(0,a : V^{p-1/2})$ *(respectively,* $r \in \mathcal{C}([0,a] : V^p) \cap H^1(0,a : V^{p-1/2}))$ *is the solution of the initial value problem (35). Furthermore, each Fourier coefficient* y_i, $i = 1, \cdots, \infty$, *is given by*

$$\begin{aligned} y_i(x_1) \quad = \quad & \sqrt{\lambda_i} \sinh(\sqrt{\lambda_i} x_1) \int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt \\ & + y_{ai} \frac{\sinh \sqrt{\lambda_i} x_1}{\sinh \sqrt{\lambda_i} a}. \end{aligned} \tag{41}$$

Theorem 11 *Let* $j, k, q \in I\!R$ *arbitrary,* $y_0 \in V^j$, $f \in \mathcal{C}([0,a] : V^k)$ *(respectively,* $f \in L^2(0,a : V^k))$ *and* $y_a \in V^q$. *Then, if* $p = \min\{j, k+1, q+1\}$, *the unique solution* $y \in \mathcal{C}([0,a] : V^{p-1}) \cap \mathcal{C}^1((0,a] : V^{p-2}) \cap \mathcal{C}^2((0,a] : V^{p-3})$ *(respectively,* $y \in \mathcal{C}([0,a] : V^{p-1}) \cap H^1(0,a : V^{p-2}) \cap H^2(0,a : V^{p-3}))$*, of the initial value problem (40) is also the unique solution of the elliptic problem*

$$\begin{cases} -\Delta y = f & \text{in } \Omega, \\ y = 0 & \text{on } \Sigma, \\ y = y_0 & \text{on } \Gamma_0, \\ y = y_a & \text{on } \Gamma_a, \end{cases} \tag{42}$$

Proof. By Theorem 1, the equation of problem (40) is satisfied in $\mathcal{C}^1([0,a] : V^{p-1})$ (respectively, in $H^1(0,a : V^{p-1})$). Then, derivating with respect to the x_1-variable we get

$$\frac{\partial y}{\partial x_1} = -\frac{dP}{dx_1}\frac{\partial y}{\partial x_1} - P\frac{\partial^2 y}{\partial x_1^2} + \frac{dr}{dx_1},$$

which is an equation with all its terms in $\mathcal{C}(0,a : V^{p-2})$ (respectively, in $L^2(0,a : V^{p-2})$) (see Theorem 1 and regularity results of Corollary 4). Then, taking into account the equation satisfied by the Hilbert-Schmidt operator P, we have that

$$P\frac{\partial^2 y}{\partial x_1^2} - P\Delta_z P\frac{\partial y}{\partial x_1} + P\Delta_z r + Pf = 0.$$

Therefore, we have the following equation in $\mathcal{C}(0,a : V^{p-1})$ (respectively, in $L^2(0,a : V^{p-1})$):

$$P\left(\frac{\partial^2 y}{\partial x_1^2} + \Delta_z y + f\right) = 0.$$

Hence, since $P(x_1) < 0$ for all $x_1 > 0$, we have that

$$-\Delta y = -\frac{\partial^2 y}{\partial x_1^2} - \Delta_z y = f \quad \text{in } \mathcal{C}(0,a : V^{p-2}) \text{ (respectively, in } L^2(0,a : V^{p-2})).$$

Finally, the boundary conditions are obviously satisfied, which concludes the proof. □

Remark 10 Once we know that the solution of (40) is the unique solution of (42) we can improve the regularity results showed in Lemma 5 and Corollary 4. For instance, if $f \in L^2(\Omega)$ and $y_0, y_a \in V^{1/2} = H_{00}^{1/2}(\mathcal{O})$, then $y \in \mathcal{C}([0,a] : V^{1/2}) \cap \mathcal{C}^1([0,a] : V^{-1/2})$ (see [7]). More regularity results for these kind of problems can be seen, for instance, in [6]. We also can improve the regularity results of Lemma 5, as showed in Theorem 12.

Remark 11 The above explicit formula (41) is not well-suited for computational purposes, since at $x = 0$ there is a multiplication $0 \cdot \infty$. Nevertheless, we can avoid this problem by developing the right hand term, obtaining the following result

Theorem 12 *Let $j, k, q \in I\!R$ arbitrary, $y_0 \in V^j$, $f \in \mathcal{C}([0,a] : V^k)$ (respectively, $f \in L^2(0,a : V^k)$) and $y_a \in V^q$. Then, if $p = \min\{j, k+1, q\}$, there exists a unique solution $y \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1((0,a] : V^{p-1})$ (respectively, $y \in \mathcal{C}([0,a] : V^{p-1}) \cap H^1(0,a : V^{p-2})$), of the initial value problem (40) (or, equivalently, problem (42)), where $r \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1[0,a] : V^{p-1}) \cap L^2(0,a : V^{p+1/2}) \cap H^1(0,a : V^{p-1/2})$ (respectively, $r \in \mathcal{C}([0,a] : V^p) \cap H^1(0,a : V^{p-1/2})$) is the solution of the initial value problem (35). Furthermore, each Fourier coefficient y_i, $i = 1, \cdots, \infty$, is given by*

$$\begin{aligned} y_i(x_1) &= r_i(x_1) - r_i(a)\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)} + y_{ai}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)} \\ &\quad + \sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{-1}{\cosh(\sqrt{\lambda_i}t)}\left(\sqrt{\lambda_i}r_i(t) - \frac{f_i(t)}{\sqrt{\lambda_i}}\right)dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} y_i(x_1) &= \frac{e^{-\sqrt{\lambda_i}x_1}}{\sqrt{\lambda_i}}\int_0^{x_1}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\ &\quad -\frac{e^{-\sqrt{\lambda_i}a}}{\sqrt{\lambda_i}}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\int_0^{a}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\ &\quad -\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sqrt{\lambda_i}}\int_{x_1}^{a}e^{-\sqrt{\lambda_i}t}f_i(t)dt \\ &\quad +\sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{1}{\sqrt{\lambda_i}}\frac{f_i(t)}{\cosh(\sqrt{\lambda_i}t)}dt \\ &\quad +y_{0i}\left(e^{-\sqrt{\lambda_i}x_1} - e^{-\sqrt{\lambda_i}a}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\right) \\ &\quad +y_{ai}\frac{\sinh\sqrt{\lambda_i}x_1}{\sinh\sqrt{\lambda_i}a}. \end{aligned}$$

Proof. See Appendix A.4. □

Example 3 Let $f \equiv 1$ in Ω. By Theorem 12 we have that the solution of problem (40) (or, equivalently, problem (42)) is

$$\begin{aligned}
y_i(x_1) &= \frac{e^{-\sqrt{\lambda_i}x_1}}{\lambda_i}\sinh(\sqrt{\lambda_i}x_1) \\
&\quad -\frac{e^{-\sqrt{\lambda_i}a}}{\lambda_i}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\sinh(\sqrt{\lambda_i}x_1) \\
&\quad -\frac{\sinh(\sqrt{\lambda_i}x_1)}{\lambda_i}\left(e^{-\sqrt{\lambda_i}x_1} - e^{-\sqrt{\lambda_i}a}\right) \\
&\quad +\frac{2\sinh(\sqrt{\lambda_i}x_1)}{\lambda_i}\left(\arctan(e^{\sqrt{\lambda_i}a}) - \arctan(e^{\sqrt{\lambda_i}x_1})\right) \\
&\quad +y_{0i}\left(e^{-\sqrt{\lambda_i}x_1} - e^{-\sqrt{\lambda_i}a}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\right) \\
&\quad +y_{ai}\frac{\sinh\sqrt{\lambda_i}x_1}{\sinh\sqrt{\lambda_i}a} \\
&= \frac{\sinh(\sqrt{\lambda_i}x_1)}{\lambda_i}\Bigg(e^{-\sqrt{\lambda_i}a} - e^{-\sqrt{\lambda_i}a}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)} \\
&\qquad + 2\arctan(e^{\sqrt{\lambda_i}a}) - 2\arctan(e^{\sqrt{\lambda_i}x_1})\Bigg) \\
&\quad +y_{0i}\left(e^{-\sqrt{\lambda_i}x_1} - e^{-\sqrt{\lambda_i}a}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\right) \\
&\quad +y_{ai}\frac{\sinh\sqrt{\lambda_i}x_1}{\sinh\sqrt{\lambda_i}a}.
\end{aligned}$$

Now, let us suppose that $\Omega = (0,a)\times(0,b)$ (that is, $\mathcal{O} = (0,b)$). Then, the eigenvalues λ_n and the eigenfunctions w_n are those given in Example 2 and

$$y(x_1,x_2) = \sqrt{\frac{2}{b}}\sum_{n=1}^{\infty} y_n \sin(\frac{n\pi}{b}x_2).$$

which can be approximated by

$$y^m(x_1,x_2) = \sqrt{\frac{2}{b}}\sum_{n=1}^{m} y_n \sin(\frac{n\pi}{b}x_2),$$

for m large enough.

For instance, if $f \equiv 1$, $y_0 \equiv 1$ and $y_a \equiv 1$, then

$$f_n = y_{0n} = y_{an} = \begin{cases} \frac{2\sqrt{2b}}{n\pi} & \text{if } n \text{ is odd,} \\ 0 & \text{if } n \text{ is even.}\end{cases}$$

Figure 3 (respectively Figure 4) shows the linear interpolation of y^{20} (respectively y^{50}), over a grid of 16 elements for the x_1-coordinate and 26 elements for the x_2-coordinate, when $a = 3$ and $b = 5$.

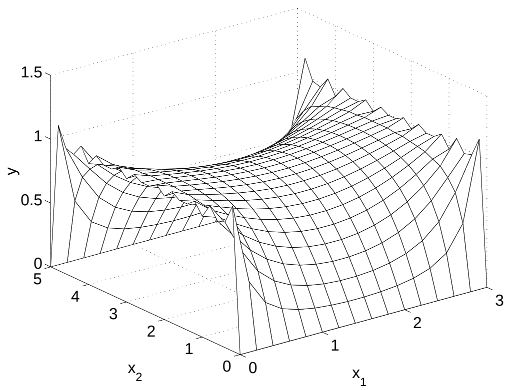

Figure 3: Graph of the function y^{20} of Example 3.

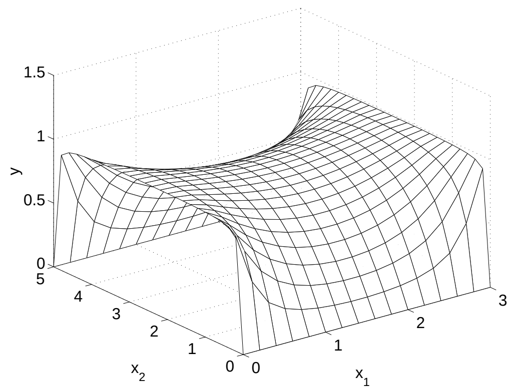

Figure 4: Graph of the function y^{50} of Example 3.

4.2.3 Neumann Condition on Γ_a

Theorem 13 *Let $j, k, q \in I\!R$ arbitrary, $y_0 \in V^j$, $f \in \mathcal{C}([0,a] : V^k)$ (respectively, $f \in L^2(0, a : V^k)$) and $y_a \in V^q$. Then, if $p = \min\{j, k+1, q+1\}$, there exists a unique solution $y \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1((0,a] : V^{p-1})$ (respectively, $y \in \mathcal{C}([0,a] : V^{p-1}) \cap H^1(0, a : V^{p-2})$), of the initial value problem*

$$\left\{ \begin{array}{l} P\dfrac{\partial y}{\partial x_1} = -y + r, \\ y(a) = -P(a)y_a + r(a), \end{array} \right. \tag{43}$$

where $r \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1[0,a] : V^{p-1}) \cap L^2(0, a : V^{p+1/2}) \cap H^1(0, a : V^{p-1/2})$ (respectively, $r \in \mathcal{C}([0,a] : V^p) \cap H^1(0, a : V^{p-1/2})$) is the solution of the initial value problem (35). Further, y is also solution of the problem

$$\left\{ \begin{array}{ll} -\Delta y = f & \textit{in } \Omega, \\ y = 0 & \textit{on } \Sigma, \\ y = y_0 & \textit{on } \Gamma_0 \\ \dfrac{\partial y}{\partial x_1} = y_a & \textit{on } \Gamma_a \end{array} \right. \tag{44}$$

and each Fourier coefficient y_i, $i = 1, \cdots, \infty$, is given by the formula of Theorem 12, changing y_a by $-P(a)y_a + r(a)$, that is

$$\begin{aligned} y_i(x_1) &= r_i(x_1) - \frac{y_{ai}}{\sqrt{\lambda_i}} \frac{\sinh(\sqrt{\lambda_i} x_1)}{\cosh(\sqrt{\lambda_i} a)} \\ &\quad + \sinh(\sqrt{\lambda_i} x_1) \int_{x_1}^{a} \frac{-1}{\cosh(\sqrt{\lambda_i} t)} \left(\sqrt{\lambda_i} r_i(t) - \frac{f_i(t)}{\sqrt{\lambda_i}} \right) dt \end{aligned}$$

or, equivalently,

$$\begin{aligned} y_i(x_1) &= \frac{e^{-\sqrt{\lambda_i} x_1}}{\sqrt{\lambda_i}} \int_0^{x_1} \cosh(\sqrt{\lambda_i} t) f_i(t) dt \\ &\quad + \frac{e^{-\sqrt{\lambda_i} a}}{\sqrt{\lambda_i}} \frac{\sinh(\sqrt{\lambda_i} x_1)}{\cosh(\sqrt{\lambda_i} a)} \int_0^{a} \cosh(\sqrt{\lambda_i} t) f_i(t) dt \\ &\quad - \frac{\sinh(\sqrt{\lambda_i} x_1)}{\sqrt{\lambda_i}} \int_{x_1}^{a} e^{-\sqrt{\lambda_i} t} f_i(t) dt \\ &\quad + \sinh(\sqrt{\lambda_i} x_1) \int_{x_1}^{a} \frac{1}{\sqrt{\lambda_i}} \frac{f_i(t)}{\cosh(\sqrt{\lambda_i} t)} dt \\ &\quad + y_{0i} \left(e^{-\sqrt{\lambda_i} x_1} + e^{-\sqrt{\lambda_i} a} \frac{\sinh(\sqrt{\lambda_i} x_1)}{\cosh(\sqrt{\lambda_i} a)} \right) \\ &\quad + y_{ai} \frac{\sinh(\sqrt{\lambda_i} x_1)}{\sinh(\sqrt{\lambda_i} a)}. \end{aligned}$$

4.2.4 Associated Optimal Control Problems

Equivalence between the solution y of (40) or, equivalently, problem (42) (respectively, (43) or, equivalently, problem (44)) and the solution of suitable control problems can be again found in a way similar to that of Section 4.1.4 (see also [7]).

4.2.5 Transparent Boundary Conditions

Theorem 14 *Let $j, k, q \in I\!R$ arbitrary, $y_0 \in V^j$, $f \in \mathcal{C}([0,a] : V^k)$ (respectively, $f \in L^2(0, a : V^k)$) and $y_a \in V^q$. Then, if $p = \min\{j, k+1, q\}$, the unique solution $y \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1((0,a] : V^{p-1})$ (respectively, $y \in \mathcal{C}([0,a] : V^{p-1}) \cap H^1(0, a : V^{p-2})$), of the initial value problem (40) (or, equivalently, problem (42)), where $r \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1[0,a] : V^{p-1}) \cap L^2(0, a : V^{p+1/2}) \cap H^1(0, a : V^{p-1/2})$ (respectively, $r \in \mathcal{C}([0,a] : V^p) \cap H^1(0, a : V^{p-1/2})$) is the solution of the initial value problem (35), is also the unique solution of the* *elliptic problem*

$$\begin{cases} -\Delta y = f & \text{in } \Omega_t, \\ y = 0 & \text{on } \Sigma, \\ -P\dfrac{\partial y}{\partial x_1} = y + r & \text{on } \Gamma_t, \\ y = y_a & \text{on } \Gamma_a \end{cases}$$

Proof. The proof is analogous to the proof given in Theorem 7. □

Theorem 15 *Let $j, k, q \in I\!R$ arbitrary, $y_0 \in V^j$, $f \in \mathcal{C}([0,a] : V^k)$ (respectively, $f \in L^2(0,a : V^k)$) and $y_a \in V^q$. Then, if $p = \min\{j, k+1, q+1\}$, the unique solution $y \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1((0,a] : V^{p-1})$ (respectively, $y \in \mathcal{C}([0,a] : V^{p-1}) \cap H^1(0,a : V^{p-2})$), of the initial value problem (43) (or, equivalently, problem (44)), where $r \in \mathcal{C}([0,a] : V^p) \cap \mathcal{C}^1[0,a] : V^{p-1}) \cap L^2(0,a : V^{p+1/2}) \cap H^1(0,a : V^{p-1/2})$ (respectively, $r \in \mathcal{C}([0,a] : V^p) \cap H^1(0,a : V^{p-1/2})$) is the solution of the initial value problem (35), is also the unique solution of the elliptic problem*

$$\begin{cases} -\Delta y = f & \text{in } \Omega_t, \\ y = 0 & \text{on } \Sigma, \\ -P\dfrac{\partial y}{\partial x_1} = y + r & \text{on } \Gamma_t, \\ \dfrac{\partial y}{\partial x_1} = y_a & \text{on } \Gamma_a \end{cases}$$

Proof. The proof is analogous to the proof given in Theorem 7. □

Remark 12 Theorems 12 and 14 (respectively, 13 and 15) imply that the solution y of (40) or, equivalently, problem (42) (respectively, (43) or, equivalently, problem (44)) is also solution of the elliptic problem

$$\begin{cases} -\Delta y = f & \text{in } \Omega_t^*, \\ y = 0 & \text{on } \Sigma, \\ y = y_0 & \text{on } \Gamma_0, \\ -P\dfrac{\partial y}{\partial x_1} = y - r & \text{on } \Gamma_t. \end{cases}$$

We point out that this problem is not well-posed, since it does not have a unique solution (for instance, $y \equiv 0$, the solution of problem (40) or, equivalently, problem (42), and the solution of problem (43) or, equivalently, problem (44) are solutions of that problem). However, from Remark 7, we can deduce that, $r(s) = \gamma|_{\Gamma_s}$, where γ is the unique solution of

$$\begin{cases} -\Delta \gamma = f & \text{in } \Omega_t^*, \\ \gamma = 0 & \text{on } \Sigma, \\ \gamma = y_0 & \text{on } \Gamma_0, \\ \dfrac{\partial \gamma}{\partial x_1} = 0 & \text{on } \Gamma_t, \end{cases}$$

which was the initial definition given in [10], [7] for function r, before deducing its associated equation.

A Appendix

A.1 Proof of Remark 5 by Dynamic Programming

We shall follow a formal way to prove the results, which shall be validated further. We shall consider Δ to be an infinitesimal and neglect terms of order Δ^2.

$$f(c,a) = \min_{v\in\mathbb{R}}\{(\frac{v^2}{\lambda_i}+c^2)\Delta + f(c+v\Delta, a-\Delta) + O(\Delta^2)\}.$$

Observe that this step is not rigorous, since the term $O(\Delta^2)$ is dependent on v. Now, expending in a Taylor series, we have

$$f(c,a) = \min_{v\in\mathbb{R}}\{(\frac{v^2}{\lambda_i}+c^2)\Delta + f(c,a) + v\frac{\partial f}{\partial c}\Delta - \frac{\partial f}{\partial a}\Delta + O(\Delta^2)\}.$$

Observe that we are assuming that f possesses partial derivatives. Finally, cancelling the common terms on both sides, dividing through by Δ, and letting $\Delta \to 0$, we obtain

$$\frac{\partial f}{\partial a} = \min_{v\in\mathbb{R}}\{c^2 + \frac{v^2}{\lambda_i} + v\frac{\partial f}{\partial c}\}.$$

It is clear that the optimal policy $u = \eta'$ is given by

$$u(c,a) = -\frac{\lambda_i}{2}\frac{\partial f}{\partial c},$$

and thus,

$$\begin{cases} \dfrac{\partial f}{\partial a} = c^2 - \dfrac{\lambda_i}{4}\left(\dfrac{\partial f}{\partial c}\right)^2, \\ f(c,0) = 0. \end{cases}$$

The above equations provide us with both the minimum value $f(c,a)$ and the optimal policy u (and, therefore, the optimal trajectory η).

Let η be the optimal trajectory solution of (13). We define μ by $\eta = c\mu$. Then, the quadratic nature of the integrand of J implies that $f(c,a) = c^2 J(\mu)$. Now, it is obvious that function $r(a) = J(\mu)$ satisfies

$$r'(a) = 1 - \lambda_i r^2(a),$$

and therefore $r = -\xi_{ii}$. Thus,

$$f(c,x_1) = -c^2\xi_{ii}(x_1) = \frac{c^2}{\sqrt{\lambda_i}}\tanh(\sqrt{\lambda_i}x_1),$$

and

$$u(x_1) = u(\eta(x_1), a-x_1) = -\frac{\lambda_i}{2}\frac{\partial f}{\partial c}(\eta(x_1), a-x_1) = \lambda_i\xi_{ii}(a-x_1)\eta(x_1).$$

On the other hand, for all $\psi \in \mathcal{U}_0$, $\psi \neq 0$, we have that $\eta + \varepsilon\psi \in \mathcal{U}_c$ and, therefore, $J(\eta) < J(\eta + \varepsilon\psi)$, which implies that

$$\int_0^a \left(\frac{1}{\lambda_i}\eta'\psi' + \eta\psi \right) dx_1 = 0.$$

Now, if we knew that η had a second derivative, integrating by parts we would obtain

$$\frac{1}{\lambda_i}\eta'(a)\psi(a) + \int_0^a \left(-\frac{1}{\lambda_i}\eta'' + \eta \right) \psi dx_1 = 0, \quad \forall\ \psi \in \mathcal{U}_0.$$

From here we suspect that both terms of the left hand side have to be zero, for all $\psi \in \mathcal{U}_0$, and therefore η is solution of (14), whose solution is

$$\eta(x_1) = \frac{\cosh(\sqrt{\lambda_i}(a - x_1))}{\cosh(\sqrt{\lambda_i}a)} c.$$

Now, it is easy to verify that, as proved above (also in a formal way)

$$\eta'(x_1) = u(x_1) = u(\eta(x_1), a - x_1) = -\frac{\lambda_i}{2}\frac{\partial f}{\partial c}(\eta(x_1), a - x_1) = \lambda_i \xi_{ii}(a - x_1)\eta(x_1).$$

Finally, let us validate the results proved in a formal way. To this end, let us show that the solution found satisfies the minimization property: Let $z \in \mathcal{U}_c$, $z \neq \eta$, and $\psi = z - \eta$. Then, it is easy to prove that

$$J(z) = J(\eta) + J(\psi) > J(\eta),$$

which concludes the proof. $\square$

A.2 Proof of (34)

The solution of $\dfrac{d\overline{r}_i}{dx_1} = \xi_{ii}\lambda_i\overline{r}_i$ is given by

$$\overline{r}_i(x_1) = \frac{C}{\cosh\left(\sqrt{\lambda_i}x_1\right)}.$$

Then, we look for our solution in the form

$$r_i(x_1) = \frac{C(x_1)}{\cosh\left(\sqrt{\lambda_i}x_1\right)},$$

with $C(x_1)$ being a function to be determined, such that (33) is satisfied. Therefore,

$$\begin{aligned} \frac{dr_i}{dx_1} &= \frac{C'(x_1)\cosh\left(\sqrt{\lambda_i}x_1\right) - \sqrt{\lambda_i}C(x_1)\sinh\left(\sqrt{\lambda_i}x_1\right)}{\left(\cosh\left(\sqrt{\lambda_i}x_1\right)\right)^2} \\ &= \frac{-1}{\sqrt{\lambda_i}}\left(\frac{\sinh\left(\sqrt{\lambda_i}x_1\right)}{\cosh\left(\sqrt{\lambda_i}x_1\right)}\right)\left(\lambda_i\frac{C(x_1)}{\cosh\left(\sqrt{\lambda_i}x_1\right)} - f_i\right), \end{aligned}$$

which implies that

$$C'(x_1) = \frac{f_i(x_1)}{\sqrt{\lambda_i}} \sinh(\sqrt{\lambda_i} x_1), \qquad \text{with } C(0) = 0.$$

Thus,

$$C(x_1) = \int_0^{x_1} \frac{\sinh(\sqrt{\lambda_i} t)}{\sqrt{\lambda_i}} f_i(t) dt. \quad \square$$

A.3 Proof of (37)

The solution of $\xi_{ii} \frac{d\overline{y}_i}{dx_1} = -\overline{y}_i$ is given by

$$\overline{y}_i(x_1) = C \sinh(\sqrt{\lambda_i} x_1).$$

Then, we look for our solution in the form

$$y_i(x_1) = C(x_1) \sinh(\sqrt{\lambda_i} x_1),$$

with $C(x_1)$ being a function to be determined, such that (36) is satisfied (we point out that $y_i(0) = r_i(0) = y_{0i}$ and, therefore, if $y_{0i} \neq 0$, the condition $|C(0)| = \infty$ must hold).

Hence,

$$\begin{aligned} \frac{dy_i}{dx_1} &= C'(x_1) \sinh(\sqrt{\lambda_i} x_1) + \sqrt{\lambda_i} C(x_1) \cosh \sqrt{\lambda_i} x_1 \\ &= \sqrt{\lambda_i} C(x_1) \cosh \sqrt{\lambda_i} x_1 - \sqrt{\lambda_i} r_i(x_1) \coth(\sqrt{\lambda_i} x_1), \end{aligned}$$

which implies that

$$C'(x_1) = -\frac{\sqrt{\lambda_i} r_i(x_1) \cosh(\sqrt{\lambda_i} x_1)}{\left(\sinh(\sqrt{\lambda_i} x_1)\right)^2}, \quad \text{with } C(a) = 0.$$

Thus

$$C(x_1) = \sqrt{\lambda_i} \int_{x_1}^{a} r_i(t) \frac{\cosh(\sqrt{\lambda_i} t)}{\left(\sinh(\sqrt{\lambda_i} t)\right)^2} dt. \quad \square$$

A.4 Proof of Theorem 12

Let us suppose that $y_a \equiv 0$ (otherwise we just have to add the additional term given by the last term of the formula (41)). Integrating by parts in (41), for $i = 1, \cdots, \infty$, we obtain

$$
\begin{aligned}
y_i(x_1) &= r_i(x_1) - r_i(a)\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)} \\
&\quad + \sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{dr_i}{dx_1}(t)\frac{1}{\sinh(\sqrt{\lambda_i}t)}dt \\
&= r_i(x_1) - r_i(a)\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)} \\
&\quad + \sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{-1}{\cosh(\sqrt{\lambda_i}t)}\left(\sqrt{\lambda_i}r_i(t) - \frac{f_i(t)}{\sqrt{\lambda_i}}\right)dt \\
&= \frac{1}{\sqrt{\lambda_i}}\frac{\int_0^{x_1}\cosh(\sqrt{\lambda_i}t)f_i(t)dt}{\cosh(\sqrt{\lambda_i}x_1)} \\
&\quad - \frac{2}{\sqrt{\lambda_i}}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(2\sqrt{\lambda_i}a)}\int_0^{a}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\
&\quad + \frac{2y_{0i}}{\cosh(\sqrt{\lambda_i}x_1)} - y_{0i}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(2\sqrt{\lambda_i}a)} \\
&\quad - \sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{\int_0^{t}\cosh(\sqrt{\lambda_i}\tau)f_i(\tau)d\tau}{\left(\cosh(\sqrt{\lambda_i}t)\right)^2}dt \\
&\quad - \sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{\sqrt{\lambda_i}y_{0i}}{\left(\cosh(\sqrt{\lambda_i}t)\right)^2}dt \\
&\quad + \sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{1}{\sqrt{\lambda_i}}\frac{f_i(t)}{\cosh(\sqrt{\lambda_i}t)}dt \\
&= \frac{1}{\sqrt{\lambda_i}}\frac{\int_0^{x_1}\cosh(\sqrt{\lambda_i}t)f_i(t)dt}{\cosh(\sqrt{\lambda_i}x_1)} \\
&\quad - \frac{2}{\sqrt{\lambda_i}}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(2\sqrt{\lambda_i}a)}\int_0^{a}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\
&\quad + \frac{y_{0i}}{\cosh(\sqrt{\lambda_i}x_1)} - 2y_{0i}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(2\sqrt{\lambda_i}a)} \\
&\quad - \frac{1}{\sqrt{\lambda_i}}e^{-\sqrt{\lambda_i}x_1}\tanh(\sqrt{\lambda_i}x_1)\int_0^{x_1}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\
&\quad + \frac{1}{\sqrt{\lambda_i}}e^{-\sqrt{\lambda_i}a}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\cosh(\sqrt{\lambda_i}a)}\int_0^{a}\cosh(\sqrt{\lambda_i}t)f_i(t)dt
\end{aligned}
$$

$$
\begin{aligned}
& -\frac{2\sinh(\sqrt{\lambda_i}x_1)}{\sqrt{\lambda_i}}\int_{x_1}^{a}\frac{\cosh(\sqrt{\lambda_i}t)}{e^{2\sqrt{\lambda_i}t}+1}f_i(t)dt \\
& -\frac{2\sinh(\sqrt{\lambda_i}x_1)y_{0i}}{e^{2\sqrt{\lambda_i}x_1}+1}+\frac{2\sinh(\sqrt{\lambda_i}x_1)y_{0i}}{e^{2\sqrt{\lambda_i}a}+1} \\
& +\sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{1}{\sqrt{\lambda_i}}\frac{f_i(t)}{\cosh(\sqrt{\lambda_i}t)}dt \\
= \quad & \frac{e^{-\sqrt{\lambda_i}x_1}}{\sqrt{\lambda_i}}\int_0^{x_1}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\
& -\frac{e^{-\sqrt{\lambda_i}a}}{\sqrt{\lambda_i}}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\int_0^{a}\cosh(\sqrt{\lambda_i}t)f_i(t)dt \\
& -\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sqrt{\lambda_i}}\int_{x_1}^{a}e^{-\sqrt{\lambda_i}t}f_i(t)dt \\
& +\sinh(\sqrt{\lambda_i}x_1)\int_{x_1}^{a}\frac{1}{\sqrt{\lambda_i}}\frac{f_i(t)}{\cosh(\sqrt{\lambda_i}t)}dt \\
& +y_{0i}\left(e^{-\sqrt{\lambda_i}x_1}-e^{-\sqrt{\lambda_i}a}\frac{\sinh(\sqrt{\lambda_i}x_1)}{\sinh(\sqrt{\lambda_i}a)}\right).
\end{aligned}
$$

Finally, the regularity results can be easily deduced from this formula. □

References

[1] BALAKRISHNAN, A. V., *Functional Analysis*, Springer-Verlag, New York, 1976.

[2] BELLMAN, R., *Introduction to the Mathematical Theory of Control Processes. Volume 1: Linear Equations and Quadratic Criteria*, Academic Press, New York and London, 1967.

[3] ANGEL, E., AND BELLMAN, R., *Dynamic Programming and Partial Differential Equations*, Academic Press, 1971.

[4] BREZIS, H., *Analyse fonctionnelle. Théorie et applications*, Masson, 1983.

[5] DUNFORD, N. AND SCHWARTZ, J. T., *Linear Operators. Part II: Spectral Theory*, Academic Press, New York, 1963.

[6] GRISVARD, P., *Elliptic Problems in Nonsmooth Domains*, Pitman, 1985.

[7] HENRY, J. AND RAMOS, A. M., Factorization of Second Order Elliptic Boundary Value Problems by Dynamic Programming. Submitted.

[8] LIONS, J.L., *Contrôle Optimal de Systèmes Gouvernés par des Équations aux Dérivées Partielles*, Dunod, 1968.

[9] LIONS, J. L. AND MAGENES, E., *Problèmes aux Limites Non Homogènes et Applications*, Vol. 1 and 2, Dunod, Paris, 1968.

[10] RAMOS, A. M., *Algunos problemas en ecuaciones en derivadas parciales relacionados con la teoría de Control.* Ph.D. Thesis, Universidad Complutense de Madrid, 1996.

[11] REID, W. T., *Riccati Differential Equations*, Academic Press, 1972.

[12] TEMAM, R., Sur l'équation de Riccati associée à des opérateurs non bornés, en dimension infinie, *Journal of Functional Analysis*, 7, pp. 85-115, 1971.

[13] WEYL, H., Das asymptotisch Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, *Math. Ann.* 71, 1911, pp 441-479.

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